Positive linear span

For now, use n-dimensional vector space $\mathbf{R}^n$. Later, wrench space and velocity twist space.

Let $\mathbf{v}$ be any non-zero vector in $\mathbf{R}^n$. Then the set of vectors

$$\{k\mathbf{v} \mid k \geq 0\} \tag{1}$$

describes a *ray*.

Let $\mathbf{v}_1$, $\mathbf{v}_2$ be non-zero and non-parallel. Then the set of positively scaled sums

$$\{k_1\mathbf{v}_1 + k_2\mathbf{v}_2 \mid k_1, k_2 \geq 0\} \tag{2}$$

is a planar cone—sector of a plane.

Generalize by defining the *positive linear span* of a set of vectors $\{\mathbf{v}_i\}$:

$$\mathrm{pos}(\{\mathbf{v}_i\}) = \{\sum k_i\mathbf{v}_i \mid k_i \geq 0\} \tag{3}$$

(The positive linear span of the empty set is the origin.)

Relatives of positive linear span

The *linear span*

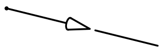
$$\mathrm{lin}(\{\mathbf{v}_i\}) = \{\sum k_i \mathbf{v}_i \mid k_i \in \mathbf{R}\} \tag{4}$$

The *convex hull*

$$\mathrm{conv}(\{\mathbf{v}_i\}) = \{\sum k_i \mathbf{v}_i \mid k_i \geq 0, \sum k_i = 1\} \tag{5}$$

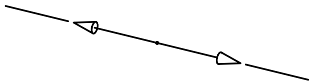
Varieties of cones in three space

1 edge

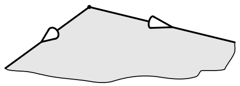


a. ray

2 edges

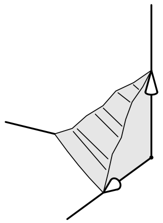


b. line

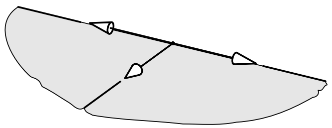


c. planar cone

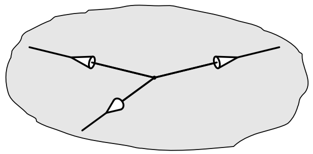
3 edges



d. solid cone

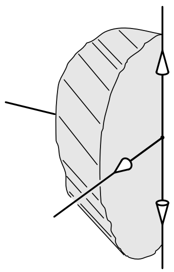


e. half plane

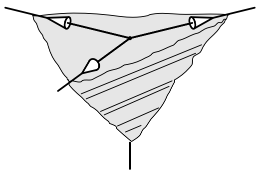


f. plane

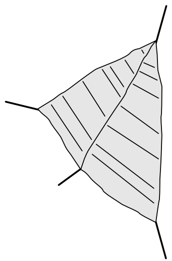
4 edges



g. wedge



h. half space



i. whole space

Spanning all of $\mathbf{R}^n$

Theorem: A set of vectors $\{\mathbf{v}_i\}$ positively spans the entire space $\mathbf{R}^n$ if and only if the origin is in the interior of the convex hull:

$$\mathrm{pos}(\{\mathbf{v}_i\}) = \mathbf{R}^n \leftrightarrow \mathbf{0} \in \mathrm{int}(\mathrm{conv}(\{\mathbf{v}_i\})) \tag{6}$$

Theorem: It takes at least $n+1$ vectors to positively span $\mathbf{R}^n$.

Two contacts

Given two frictionless contacts $\mathbf{w}_1$ and $\mathbf{w}_2$, total wrench is the sum of possible positive scalings of $\mathbf{w}_1$ and $\mathbf{w}_2$:

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2; k_1, k_2 \geq 0 \tag{12}$$

i.e. the positive linear span $\mathrm{pos}(\{\mathbf{w}_1, \mathbf{w}_2\})$.

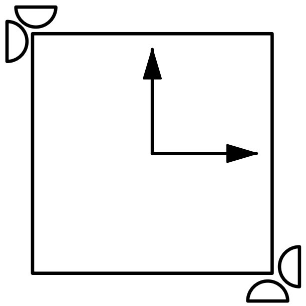
Generalizing:
Theorem: If a set of frictionless contacts on a rigid body is described by the contact normals $\mathbf{w}_i = (\mathbf{c}_i, \mathbf{c}_{0i})$ then the set of all possible wrenches is given by the positive linear span $\mathrm{pos}(\{\mathbf{w}_i\})$.

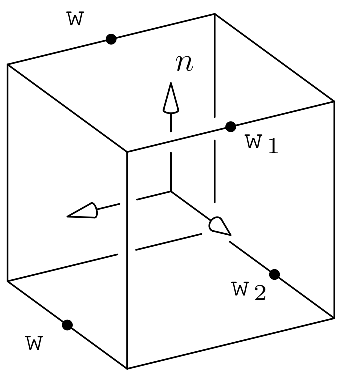
Force closure

Definition: **Force closure** means that the set of possible wrenches exhausts all of wrench space.

It follows from theorem ? that a frictionless force closure requires at least 7 contacts. Or, since planar wrench space is only three-dimensional, frictionless force closure in the plane requires at least 4 contacts.

Example wrench cone





Construct unit magnitude force at each contact.

Write screw coords of wrenches.

Take positive linear span.

Exhausts wrench space?