Motion and Manipulation

Kinematics: Orientations
Axis-Angle Rotation

- Any rotation $\Phi$ of a rigid body with fixed point $O$ about an axis $\hat{u}$ can be decomposed into three rotations about three (non-planar) axes.

- The final orientation of a rigid body after a finite number of rotations is equivalent to that obtained after a unique rotation about a unique axis.
Axis-Angle Rotation

Body/moving frame \( M \) rotates by an angle \( \Phi \) about a line through the origin with direction given by the global normalized vector \( \hat{u}=(u_1,u_2,u_3)^T \), so

\[
\sqrt{u_1^2 + u_2^2 + u_3^2} = 1
\]
Angle-Axis Rotation

Rotation matrix $R$ that maps coordinates in $M$ to coordinates in $F$ is given by

$$R = I \cos \Phi + \hat{u} \hat{u}^T \text{vers} \Phi + \tilde{u} \sin \Phi$$

where $I$ is the identity matrix and

$$\text{vers} \Phi = 1 - \cos \Phi = 2 \sin^2 \frac{\Phi}{2}$$

and

$$\tilde{u} = \begin{pmatrix}
0 & -u_3 & u_2 \\
- u_3 & 0 & - u_1 \\
- u_2 & u_1 & 0
\end{pmatrix}$$
Note that the matrix $\tilde{u}$ is skew-symmetric: it satisfies

$$\tilde{u}^T = -\tilde{u}$$

Observe also that

$$\hat{u} \hat{u}^T = \begin{pmatrix}
    u_1^2 & u_1 u_2 & u_1 u_3 \\
    u_2 u_1 & u_2^2 & u_2 u_3 \\
    u_3 u_1 & u_3 u_2 & u_3^2
\end{pmatrix}$$
Angle-Axis Rotation

After substitutions, we obtain the rotation matrix

\[
R = \begin{pmatrix}
    u_1^2 \text{vers} \Phi + \cos \Phi & u_1 u_2 \text{vers} \Phi - u_3 \sin \Phi & u_1 u_3 \text{vers} \Phi + u_2 \sin \Phi \\
    u_2 u_1 \text{vers} \Phi + u_3 \sin \Phi & u_2^2 \text{vers} \Phi + \cos \Phi & u_2 u_3 \text{vers} \Phi - u_1 \sin \Phi \\
    u_3 u_1 \text{vers} \Phi - u_2 \sin \Phi & u_3 u_2 \text{vers} \Phi + u_1 \sin \Phi & u_3^2 \text{vers} \Phi + \cos \Phi
\end{pmatrix}
\]
Exercise

The unit cube is rotated by $\pi/4$ about the line through its corners A and G. What are the coordinates of the cube’s corners after the rotation?
Angle-Axis Rotation

If the transformation matrix $R$ for the rotation is given, then we can also obtain the rotation angle $\Phi$ and axis $\mathbf{u}$ using

$$
cos \Phi = \frac{1}{2} \left( tr(R) - 1 \right)
$$

$$
\mathbf{u} = \frac{1}{2 \sin \Phi} \left( R - R^T \right)
$$

where

$$
tr\left( \begin{pmatrix} r_{11} & \ldots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \ldots & r_{nn} \end{pmatrix} \right) = \sum_{i=1}^{n} r_{ii}
$$
Exercise

A body frame $M$ undergoes Euler rotations $\varphi=\pi/6$ (about the global/local $z$-axis), then by $\theta=\pi/4$ (about the local $x$-axis), and then by $\psi=\pi/3$ (about the local $z$-axis). Determine angle and axis of rotation.

Recall:

$$R = \begin{pmatrix}
\cos \varphi \cos \psi - \cos \theta \sin \varphi \sin \psi & - \cos \varphi \sin \psi - \cos \theta \cos \psi \sin \varphi & \sin \theta \sin \varphi \\
\cos \psi \sin \varphi + \cos \theta \cos \varphi \sin \psi & - \sin \varphi \sin \psi + \cos \theta \cos \varphi \cos \psi & - \cos \varphi \sin \theta \\
\sin \theta \sin \varphi & \sin \theta \cos \varphi & \cos \theta
\end{pmatrix}$$
Euler Parameters

Given rotation angle $\Phi$ and axis represented by global unit vector $\hat{u}=(u_1,u_2,u_3)^T$, the Euler parameters $(e_0,e_1,e_2,e_3)$ are given by

- $e_0 = \cos \frac{\Phi}{2}$
- $e_1 = u_1 \sin \frac{\Phi}{2}$
- $e_2 = u_2 \sin \frac{\Phi}{2}$
- $e_3 = u_3 \sin \frac{\Phi}{2}$
Euler Parameters to Matrix

The transformation matrix $R$ corresponding to the Euler parameters $(e_0, e_1, e_2, e_3)$ is

$$
R = \begin{bmatrix}
    e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_0e_2 + e_1e_3) \\
    2(e_0e_3 + e_1e_2) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\
    2(e_1e_3 - e_0e_2) & 2(e_0e_1 + e_2e_3) & e_0^2 - e_1^2 - e_2^2 + e_3^2
\end{bmatrix}
$$
Matrix to Euler Parameters

If the transformation matrix $R$ for the rotation is given then we can also obtain the Euler parameters using

$$e_0^2 = \frac{1}{4}(\text{tr}(R) + 1)$$
$$\tilde{e} = \frac{1}{4e_0} (R - R^T)$$

where also

$$\tilde{e} = \begin{pmatrix}
0 & -e_3 & e_2 \\
e_3 & 0 & -e_1 \\
-e_2 & e_1 & 0
\end{pmatrix}$$
Exercise

Determine the Euler parameters for a rotation angle $\Phi = \pi/3$ and axis given by

$$\hat{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{pmatrix}$$

Use the Euler parameters to find the transformation matrix $R$ and then use $R$ to get the Euler parameters back again.
Matrix to Euler Parameters

We can identify the Euler parameters \((e_0, e_1, e_2, e_3)\) from transformation matrix \(R = (r_{ij})_{1 \leq i, j \leq 3}\) using:

\[
\begin{align*}
e_0 &= \pm \frac{1}{2} \sqrt{1 + r_{11} + r_{22} + r_{33}} \\
e_1 &= \frac{1}{4} \frac{r_{32} - r_{23}}{e_0} \\
e_2 &= \frac{1}{4} \frac{r_{13} - r_{31}}{e_0} \\
e_3 &= \frac{1}{4} \frac{r_{21} - r_{12}}{e_0}
\end{align*}
\]
Quaternions

• Extension of complex numbers

\[ q = a + bi + cj + dk \]

where \( a, b, c, \) and \( d \) are reals, and \( i, j, k \) are the quaternion units

• \( a \) is referred to as the \textit{scalar part}
• \( bi + cj + dk \) is the \textit{vector part}

• A \textit{pure quaternion} is a quaternion with \( a=0 \)
Quaternion Arithmetic

- Identities

\[ i^2 = j^2 = k^2 = ijk = -1 \]

from which it follows that

\[
\begin{align*}
ij &= k, \\
ji &= -k \\
jk &= i, \\
kj &= -i \\
ki &= j, \\
ik &= -j \\
\end{align*}
\]

Quaternion multiplication is not commutative
Hamiltonian Product

\[(a_1 + b_1i + c_1j + d_1k)(a_2 + b_2i + c_2j + d_2k) =\]

\[a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2\]
\[+ (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i\]
\[+ (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j\]
\[+ (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k\]
Conjugate and Norm

- **Conjugate** $q^* = a - bi - cj - dk$

- $(pq)^* = q^* p^*$ (similar to inverse of matrix multiplication)

- **Norm** $|q|$ with $q = a + bi + cj + dk$ is given by

  $$|q|^2 = qq^* = a^2 + b^2 + c^2 + d^2$$

- A unit quaternion is a quaternion of norm 1
Inverse of a Quaternions

- Inverse $q^{-1}$ of $q$ given by

$$q^{-1} = \frac{q^*}{|q|^2}$$

- For a unit quaternion: $q^{-1} = q^*$
Quaternions and Rotations

Consider the rotation of a vector $p=(p_1,p_2,p_3)^T$ by an angle $\Phi$ about the line through the origin with (normalized) direction vector $\hat{u}=(u_1,u_2,u_3)^T$

- Rotation can be represented by a unit quaternion

$$q = e^{\frac{\Phi}{2}(u_1i+u_2j+u_3k)}$$

which by Euler’s formula corresponds to

$$q = \cos \frac{\Phi}{2} + (u_1i + u_2j + u_3k) \sin \frac{\Phi}{2}$$
Quaternions and Rotations

- Represent the vector $p$ by the pure quaternion

$$p = p_1 i + p_2 j + p_3 k$$

- The image $p'$ of $p$ after rotation can be obtained by computing the conjugation of $p$ by $q$

$$p' = qpq^{-1}$$

which, since $q$ is a unit quaternion, equals

$$p' = qpq^*$$
Exercise

- Rotation by $2\pi/3$ about the line $x=\lambda(1,1,1)$; determine the image of $(1,0,0)$
Fixed and Moving Frames

If a rotation represented by a quaternion $q_1$ is followed by a rotation represented by a quaternion $q_2$ then the resulting composition is represented by a quaternion $q$ with

- $q = q_2 q_1$ if $q_2$ was about a fixed axis
- $q = q_1 q_2$ if $q_2$ was about a moving axis
Exercise

Initially M=F. Determine the coordinates with respect to F of the point (0,1,2) with respect to M if a rotation by \( \pi/2 \) about \( f_3 \) is followed by

- a rotation by \( \pi/2 \) about \( f_1 \)
- a rotation by \( \pi/2 \) about \( m_1 \)