Lecture VII: Soft-Body Physics
Soft Bodies

- Realistic objects are not purely rigid.
  - Good approximation for “hard” ones.
  - ...approximation breaks when objects break, or deform.

- Generalization: soft (deformable) bodies
  - Deformed by force: car body, punched or shot at.
  - Prone to stress: piece of cloth, flag, paper sheet.
  - Not solid: snow, mud, lava, liquid.

Grinspun et al. “Discrete Shells”

http://www.games73.com/media.games73.com/files/2012/05/Crysis-3-soft-physics-demo-thumb-610x239.jpg

http://i.huffpost.com/gen/1480563/images/o-DISNEY-facebook.jpg
Elasticity

- Forces may cause object deformation.

- **Elasticity**: the tendency of a body to return to its original shape after the forces causing the deformation cease.
  - Rubbers are highly elastic.
  - Metal rods are much less.
Continuum Mechanics

- A deformable object is defined by \textit{rest shape} and \textit{material parameters}.
- Deformation map: \( f(\mathbf{x}) \) of every material point \( \mathbf{x} \).
- \( f: \mathbb{R}^d \rightarrow \mathbb{R}^d \). \( d \): dimension (mostly \( d = 2,3 \)).
- Relative displacement field: \( f(\mathbf{x}) = \mathbf{x} + u(\mathbf{x}) \).
Local Deformation

• Taylor series:
  \[ f(\tilde{x} + \Delta \tilde{x}) \approx f(\tilde{x}) + J_f \Delta \tilde{x} \]

• 1st-order linear approximation.

• As \( f(\tilde{x}) = \tilde{x} + u(\tilde{x}) \), we get:
  \[
  \tilde{x} + \Delta \tilde{x} + u(\tilde{x} + \Delta \tilde{x}) \approx \tilde{x} + u(\tilde{x}) + J_f \Delta \tilde{x} \Rightarrow \\
  u(\tilde{x} + \Delta \tilde{x}) \approx u(\tilde{x}) + (J_f - I_{d \times d}) \Delta \tilde{x}
  \]

• The Jacobians: \( J_f = \left( \frac{\partial f}{\partial x} \right), J_u = \left( \frac{\partial u}{\partial x} \right) = J_f - I_{d \times d} \).
Stretch and Compression

• How much an object locally stretches or compresses in each direction.

• New length:

\[
|\Delta f|^2 = |f(\vec{x} + \Delta \vec{x}) - f(\vec{x})|^2 \approx |J_f \Delta \vec{x}|^2 \\
= \Delta \vec{x}^T \star (J_f^T J_f) \star \Delta \vec{x}
\]

• Stretch: relative change in length:

\[
\frac{|\Delta f|^2}{|\Delta x|^2} \approx \frac{\Delta \vec{x}^T \star (J_f^T J_f) \star \Delta \vec{x}}{\Delta \vec{x}^T \star \Delta \vec{x}}
\]

Rigid-Body Deformation

• Transformation:
  \[ f(\vec{x}) = R\vec{x} + T \]
  - \( R \): rotation (constant)
  - \( T \): translation.

• \( J_f = R \), and then \( J_f^T J_f = I \).
  - No stretch!
Cauchy-Green Deformation Tensor

- $(J_f^T J_f)_{d \times d}$ is the (right) Cauchy-Green tensor.

- **Interpretation**: for direction $\hat{d}$, we get:
  \[ |\nabla_{\hat{d}} (f(x))|^2 \approx \hat{d}^T \cdot (J_f^T J_f) \cdot \hat{d} \]

- **In words**: the CG tensor measures the ratio of change in squared length in a direction.
The Green-Lagrange Strain Tensor

• Measures the deviation from rigidity:
  \[ E_{3\times3} = \frac{1}{2} (J_f^T J_f - I) \]

• Gives “0” for rotations.
  • Rather than “1” in the CG tensor.

• In deformation field terms \((J_f = J_u + I_{d\times d})\):
  \[ E = \frac{1}{2} (J_u^T J_u + J_u + J_u^T) \]
Strain

• The fractional deformation $\varepsilon = \Delta L / L$
  • Dimensionless (a ratio).
  • How much a deformation differs from rigidity in a given direction:
    • Negative: compression
    • Zero: rigid
    • Positive: stretch

• In 1D: $\frac{|\Delta f|}{|\Delta x|} = \frac{\Delta L + L}{L} = 1 + \varepsilon$
GL Tensor and Strain

- For Strain $\epsilon = \Delta L/L$ in (unit length) direction $\hat{d}$:
  \[
  \hat{d}^T E \hat{d} = \epsilon + \frac{1}{2} \epsilon^2
  \]

- **Problem:** the GL strain tensor is nonlinear in the deformation $f$ (or the deformation field $u$).

- **Approximation:** the infinitesimal strain tensor:
  \[
  \epsilon = \frac{1}{2} \left( J_u^T J_u + J_u + J_u^T \right) \approx \frac{1}{2} \left( J_u + J_u^T \right)
  \]
  Where $\hat{d}^T E \hat{d} \approx \epsilon$.  

\[ E_{3\times3} = \frac{1}{2} (J_f^T J_f - I) \]
The Infinitesimal Strain Tensor

• A.K.A. Cauchy’s strain tensor:

\[ \mathbf{\varepsilon} = \frac{1}{2} (J_u + J_u^T) = \frac{1}{2} (J_f + J_f^T) - I \]

• So that for unit direction \( \hat{d} \):

\[ \mathbf{\varepsilon}_d = \hat{d}^T \mathbf{\varepsilon} \hat{d} \]

• The Strain tensor is not rotation invariant!

• Good for small deformations \( |J_u| \ll 1 \).
Poisson’s ratio

• Strain in one direction usually causes compression in another.

• **Poisson’s ratio**: the ratio of transversal to axial strain:

\[
\nu = - \frac{d \text{ [transversal strain]}}{d \text{ [axial strain]}}
\]

• Equals 0.5 in **perfectly incompressible** material.

• If the force is applied along \( x \):

\[
\nu = - \frac{de_y}{d\epsilon_x} = - \frac{de_z}{d\epsilon_x}
\]
Poisson’s’s ratio

• Example of a cube of size $L$.

• Average strain in each direction: $\nu \approx \frac{\Delta L'}{\Delta L}$
  - Approximate, because true for small elements and deformation.
Stress

- **Magnitude** of applied force per area of application.
  - large value $\Leftrightarrow$ force is large or surface area is small
- **Pressure measure** $\vec{\sigma}$.
- **Unit**: Pascal: $[Pa] = \left[ \frac{N}{m^2} \right]$

- Example: gravity stress on plane:
  \[
  \vec{\sigma} = \frac{mg}{(\pi r^2)}
  \]
The Linear Stress Tensor

• Measuring stress for each (unit) direction $\hat{n}$ in an infinitesimal volume element:

$$\sigma(\hat{n}) = \begin{pmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \hat{n} = \sigma\hat{n}$$

• Note that $\sigma\hat{n}$ is not necessarily parallel to $\hat{n}$!

• $\sigma\hat{n} = \langle \sigma\hat{n}, \hat{n} \rangle \hat{n} + \tau$

  - normal stress
  - shear stress
Body Material

• The amount of stress to produce a strain is a property of the material.

• **Isotropic materials**: same in all directions.

• **Modulus**: a ratio of stress to strain.
  • Usually in a linear direction, along a planar region or throughout a volume region.
    • Young’s modulus, Shear modulus, Bulk modulus
  • Describing the material reaction to stress.

http://openalea.gforge.inria.fr/doc/vplants/mechanics/doc/_build/html/_images/system_geometry7.png
Young’s Modulus

• Defined as the ratio of linear stress to linear strain:

\[ Y = \frac{\text{linear stress}}{\text{linear strain}} = \frac{F/A}{\Delta L/L} \]

• Example:
Shear modulus

• The ratio of planar stress to planar strain:

\[ S = \frac{\text{planar stress}}{\text{planar strain}} = \frac{F/A}{\Delta L/L} \]

• Example:
Bulk modulus

- The ratio of **volume stress** to **volume strain** (inverse of compressibility):

\[ B = \frac{\text{volume stress}}{\text{volume strain}} = \frac{\Delta P}{\Delta V/V} \]

- Example

\[ P = \frac{F}{A} \]

\[ P + \Delta P \]

\[ F + \Delta F \]
Linear Elasticity

• **Stress** and **strain** are related by Hooke’s law
  • Remember $F = -k \Delta x$?

• Reshape tensors to vector form:
  • $\bar{\sigma} = (\sigma_{xx}, \sigma_{xy}, \cdots, \sigma_{zz})$, and similarly for $\bar{\varepsilon}$.
  • Then the **stiffness tensor** $C_{9 \times 9}$ holds:
    $$\bar{\sigma} = C \bar{\varepsilon}$$

![Image](https://people.eecs.berkeley.edu/~sequin/CS184/TOPICS/SpringMass/Spring_mass_2D.GIF)

Hyperelastic Materials

• Seek to return to their “rest shape”.
  • Have a potential deformation energy

• Spring energy: $E_S = \frac{1}{2} k \|x - x_0\|^2$.

• **Underlying assumption**: deformation energy is not path-dependent!
Linear Elasticity Energy

• One Possibility is: $E = \frac{1}{2} \int_T \langle \bar{\sigma}, \bar{\epsilon} \rangle dV$.
  • Possibilities depend on the type of Stress\Strain tensors to use.
  • This one is popular for linear elasticity with FEM.

• We get:

$$E = \frac{1}{2} \int_T \bar{\epsilon} C \bar{\epsilon} dV$$
Dynamic Elastic Materials

• For every point \( q \), The PDE is given by

\[
\rho \frac{d^2 \vec{u}}{dt^2} = \nabla \cdot \sigma + \vec{F}
\]

• \( \rho \): the density of the material.

• \( \nabla \cdot \sigma = (\partial / \partial x, \partial / \partial y, \partial / \partial z) \cdot \sigma \) is the divergence of the stress tensor (modeling internal forces).

• \( \vec{F} \): external body forces (per point)

• Generalized Newton’s 2\textsuperscript{nd} law!
  • Remember \( F = ma \)?
  • Similar, in elasticity language.