Lecture III: Rigid-Body Physics
Rigid-Body Motion

• Previously: Point dimensionless objects moving through a trajectory.

• Today: Objects with dimensions, moving as one piece.
Rigid-Body Kinematics

• Objects as sets of points.
• Relative distances between all points are invariant to rigid movement.
• Movement has two components:
  • Linear trajectory of a central point ("translation").
  • Relative rotation around the point ("rotation").
Rotational Motion

• $\vec{P}$ a point on to the object.
• $\vec{C}$ is the center of rotation.
• Distance vector $\vec{r} = \vec{P} - \vec{C}$.
  • $r = ||\vec{r}||$: distance.
• Object rotates $\Leftrightarrow P$ travels along a circular path.
• Unit-length axis of rotation: $\hat{e}$.
  • 2D: $\hat{e} = \hat{z}$ (“out” from the screen).
  • “Positive” Rotation: counterclockwise.
    • right-hand rule.

$$\vec{r} \times \frac{d\vec{P}}{dt} \parallel \hat{e}$$
Rigid-Body Kinematics

- Object coordinate system.
- Describing rigid-body motion:
  - Moving origin
  - Rotating orientation.
- Both defined w.r.t. another frame!
  - i.e., the canonical “world coordinates”.

[Diagram of a 3D coordinate system with axes labeled X, Y, Z, and O, C(t) labeled to indicate a moving origin.]
Representing Orientation

- Object axis system (rows):
  \[
  R_o(t) = \begin{pmatrix}
  \hat{x}_o(t) \\
  \hat{y}_o(t) \\
  \hat{z}_o(t)
  \end{pmatrix}.
  \]
  - Changes with time $t$.
  - An orthonormal (rotation matrix).
- Rotations are orientations!
- Canonical ($t = 0$) axes: can be arbitrary.
  - The canonical choice $R_o(0) = I_{3\times3}$ is default ("world coordinates").
Axis-Angle Representation

• Every rotation can be represented by an axis $\hat{e}$ and a angle $\theta$.

• Angle-axis $\rightarrow$ rotation matrix:
\[
R = I + \sin \theta \, K + (1 + \cos \theta)K^2
\]

• $K = [\hat{e}]_x = \begin{pmatrix}
0 & -\hat{e}_z & \hat{e}_y \\
\hat{e}_z & 0 & -\hat{e}_x \\
-\hat{e}_y & \hat{e}_x & 0
\end{pmatrix}$

• The cross-product matrix: $Kv = \hat{e} \times v$

• Rotation matrix $\leftarrow$ angle axis:
\[
\theta = \arccos \left( \frac{\text{tr}(R) - 1}{2} \right)
\]
\[
\hat{e} = \frac{1}{2 \sin \theta} \begin{pmatrix}
R_{32} - R_{23} \\
R_{13} - R_{31} \\
R_{21} - R_{12}
\end{pmatrix}
\]

• Not unique! Why?
Quaternions

- Quaternions: \( q = (r, \vec{v}) \)
  - Real part: scalar \( r \).
  - Imaginary part: vector \( \vec{v} \in \mathbb{R}^3 \).
- Unit quaternions: \( |q|^2 = r^2 + |\vec{v}|^2 = 1 \).
- Quaternion multiplication:
  \[
  p \cdot q = (r_p r_q - \langle \vec{v}_p, \vec{v}_q \rangle, r_q \vec{v}_p + r_p \vec{v}_q + \vec{v}_p \times \vec{v}_q)
  \]
- Not commutative! \( p \cdot q \neq q \cdot p \).
- Commutative iff \( \vec{v}_p \times \vec{v}_q = 0 \) (parallel vector parts).
- Inverse and conjugate: \( \bar{p} = (r, -\vec{v}), p^{-1} = \frac{\bar{p}}{|p|^2} \).
Quaternions as rotations

• Rotations as unit quaternions:

\[ q = \left( \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \hat{e} \right) \]

• Vectors \( \vec{a} \in \mathbb{R}^3 \Leftrightarrow \) imaginary quaternions \((0, \vec{a})\).

• Rotating vector \( \vec{a} \) by \( \theta, \hat{e} \) into vector \( \vec{b} : \vec{b} = q\vec{a}q^{-1} \).

• Rotation composition: subsequent multiplication:

  • Rotation of \( q \) and then \( p \): \( \vec{c} = pbp^{-1} = pq\vec{a}q^{-1}p^{-1} = pq\vec{a}(pq)^{-1} = s\vec{a}s^{-1} \)

  • Where \( s = pq \).

• Quaternions \(\Leftrightarrow\) rotation matrices with same axis-angle.
  • Mutual conversion: a bit technical.
Special case: 2D

• Axis is always $\hat{e} = \hat{z}$.
• Quaternion $q = \left(\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \hat{z}\right)$ reduces to complex number $c = e^{i\theta/2}$.
• Rotation of (complex) vector: $\vec{b} = c^2 \vec{a}$.
• Remember $e^{i\pi} = -1$?
  • Rotation by $\pi$! (reflection through point).
Key-framing

• Interpolating between orientation quaternions $q(t)$ and $q(t + \Delta t)$?

• No single option!
  • Not even in 2D…

• **Shortest** rotation: $\Delta q(t) = q(t + \Delta t) q(t)^{-1}$

• Can be applied: $q(t + \Delta t) = \Delta q(t) q(t)$.

• What happens continuously?

Instantaneous Rotation

• A body goes through orientations \( q(t) \) (quaternion).
• The change of orientation, or derivative \( \frac{dq}{dt} \):
  \[
  \frac{dq}{dt} = \frac{1}{2} (0, \vec{\omega}) q
  \]
• What is \( \vec{\omega} \)?
Angular Velocity

• The speed of rotation around the rotation axis:
  \[ \vec{\omega}(t) = \omega(t)\hat{e}(t). \]

• The angular velocity vector is collinear with the rotation axis:

• unit is \( \text{rad/sec} \).

• \( \hat{e}(t) \): the instantaneous axis of rotation.
Angular Acceleration

- **Angular acceleration**: the rate of change of the angular velocity:

\[ \hat{\alpha} = \frac{d\vec{\omega}}{dt} \]

- Paralleling definition of linear acceleration.

- Unit is $rad/s^2$
Tangential and Angular Velocities

• Every point moves with the same angular velocity.
  • Direction of vector: \( \vec{e} \).

• Tangential velocity vector:

\[
\vec{v} = \vec{\omega} \times \vec{r} = \frac{d\vec{r}}{dt}
\]

Or:

\[
\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}
\]

• \( \omega = \frac{v}{r} \) (abs. values)
Decomposing Movement

• Define original (object) coordinate system and orientation $\hat{c}(0), q(0)$.

• Position in world coordinates:
  
  \[
  \hat{x}(t) = \hat{c}(t) + \hat{r}(t) = \hat{c}(t) + q(t)\hat{r}(0)q^{-1}(t).
  \]

  • $\hat{r} = \hat{x} - \hat{c}$.

• Combined velocity $\hat{v}(t)$:

  • Linear (translational) component:
    
    \[
    \hat{v}^\parallel(t) = \frac{d\hat{c}(t)}{dt}
    \]

  • Rotational component:
    
    \[
    \hat{v}^\perp(t) = \frac{d\hat{r}(t)}{dt} = \vec{\omega}(t) \times \hat{r}(t).
    \]

• But, are these always distinguishable?

  • Yes!

  • $\hat{r}(t)$ only depends on $q(t)$ and $\hat{r}(0)$ and not on $\hat{c}(t)$.
Combined Movement

• Position of every point

\[ \vec{p}(t) = \vec{p}_0 + \int_0^t \left( \vec{v}^\perp(t) + \vec{v}^\parallel(t) \right) dt \]

• \( \vec{v}^\parallel(t) \): translation of the object axis system.

• \( \vec{v}^\perp(t) = \vec{\omega}(t) \times \vec{r}(t) \): obj. axis system rotation around its “origin” \( \vec{c}(t) \).

• Question: how does the choice of obj. system (axes+origin) matter?
Invariance of Angular Velocity

• \( \dot{\vec{v}}(t) = \frac{d\vec{v}(t)}{dt} = \vec{\omega}(t) \times \vec{r}(t) \)

• What is the angular velocity \( \vec{\omega} \) for all points of the object?
  • The same! Since \( \vec{q} \) is the same:
    
    \[
    \frac{d\vec{r}(t)}{dt} = \frac{d\vec{q}(t)}{dt} \vec{r}(0) \vec{q}^{-1}(t) + \vec{q}(t) \vec{r}(0) \frac{d\vec{q}^{-1}(t)}{dt} = \\
    \frac{1}{2} (0, \vec{\omega}) \vec{r}(t) + \frac{1}{2} \vec{r}(t)(0, -\vec{\omega}) = (0, \vec{\omega}(t) \times \vec{r}(t))
    \]

• How does the choice of center \( \vec{c}(0) \) matters for \( \vec{\omega} \)?
  • Or the choice of \( \vec{r}(0) \).
  • And why?

• **Note:** \( \vec{\omega} \) is still represented in a (global, stationary, maybe canonical) axis system!

• Another world system = another representation = “different” \( \vec{\omega} \).
Mass

• The **measure** of the amount of matter in the **volume** of an object:

\[ m = \int_{V} \rho \, dV \]

• \( \rho \) : the **density** of each point the object volume \( V \).
• \( dV \) : the **volume element**.

• **Equivalently**: a measure of resistance to motion or change in motion.
Mass

• For a 3D object, mass is the integral over its volume:

\[ m = \int \int \int \rho(x, y, z) \, dx \, dy \, dz \]

• For uniform density \((\rho \text{ constant})\):

\[ m = \rho \cdot V \]
Center of Mass

• The center of mass (COM) is the “average” point of the object, weighted by density:

\[
\overrightarrow{COM} = \frac{1}{m} \int_V \rho \cdot \vec{p} \, dV
\]

• \( \rho \): point coordinates.

• Point of balance for the object.

• Uniform density: COM \( \Leftrightarrow \) centroid.
Center of Mass of System

- A system of bodies has a mutual center of mass:

$$COM = \frac{1}{m} \sum_{i=1}^{n} m_i \vec{p}_i$$

- \(m_i\): mass of each body.
- \(\vec{p}_i\): location of individual COM.
- \(m = \sum_{i=1}^{n} m_i\).

- **Example**: two spheres in 1D

$$x_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$
Center of Mass

- Quite easy to determine for primitive shapes

- What about complex surface based models?
Dynamics

• The **centripetal force** creates curved motion.

• In the direction of (negative) $\hat{r}$
  • Object is **in orbit**.

• Constant force $\Leftrightarrow$ circular rotation with constant **tangential velocity**.
  • Why?
Tangential & Centripetal Accelerations

• Tangential acceleration $\vec{a}$ holds:

$$\vec{a} = \vec{a} \times \vec{r}$$

• cf. velocity equation $\vec{v} = \vec{\omega} \times \vec{r}$.

• The centripetal acceleration drives the rotational movement:

$$\vec{a}_n = \frac{v^2}{r} \hat{r} = -\omega^2 \hat{r}.$$ 

• What is the “centrifugal” force?
Angular Momentum

- **Linear motion** ➔ linear momentum: \( \vec{p} = m \vec{v} \).
- **Rotational motion** ➔ angular momentum about any fixed relative point (to which \( \vec{r} \) is measured):

\[
\vec{L} = \int_V (\vec{r} \times \vec{p}) \rho dV
\]

- Unit is \( N \cdot m \cdot s \)
- \( \rho dV = dm \) (mass element)

- Angular momentum is **conserved**!
  - Just like the linear momentum.
- **Caveat:** conserved w.r.t. the same axis system.
Angular Momentum

• Plugging in angular velocity:

\[(\mathbf{r} \times \mathbf{p})dV = (\mathbf{r} \times \mathbf{v})dm = (\mathbf{r} \times (\mathbf{\omega} \times \mathbf{r}))dm\]

• Integrating, we get:

\[\mathbf{L} = \int_V \mathbf{r} \times (\mathbf{\omega} \times \mathbf{r})dm\]

• **Note:** The angular momentum and the angular velocity are not generally collinear!
The inertia Tensor

- Define: \( \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) and \( \vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \)

- For a single rotating body: the angular velocity is **constant**.

- We get:

\[
\vec{L} = \int_V \vec{r} \times (\vec{\omega} \times \vec{r}) \, dm = \int V \begin{pmatrix} (y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z \\ -yx\omega_x + (z^2 + x^2)\omega_y - yz\omega_z \\ -zx\omega_x - zy\omega_y + (x^2 + y^2)\omega_z \end{pmatrix} \, dm = \\
\int V \begin{pmatrix} (y^2 + z^2) \\ -yx \\ -zx \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \, dm = \\
\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.
\]

- **Note:** replacing integral with a (constant) matrix operating on a vector!
The Inertia Tensor

• The **inertia tensor** \( I \) only depends on the **geometry** of the object and the **relative axis system** (often, COM with principal axes):

\[
\begin{align*}
I_{xx} &= \int (y^2 + z^2) \, dm \\
I_{yy} &= \int (z^2 + x^2) \, dm \\
I_{zz} &= \int (x^2 + y^2) \, dm \\
I_{xy} &= I_{yx} = \int (xy) \, dm \\
I_{xz} &= I_{zx} = \int (xz) \, dm \\
I_{yz} &= I_{zy} = \int (yz) \, dm
\end{align*}
\]
The Inertia Tensor

• Compact form:

\[
I = \begin{bmatrix}
\int (y^2 + z^2)dm & -\int (xy)dm & -\int (xz)dm \\
-\int (xy)dm & \int (z^2 + x^2)dm & -\int (yz)dm \\
-\int (xz)dm & -\int (yz)dm & \int (x^2 + y^2)dm
\end{bmatrix}
\]

• The diagonal elements are called the (principal) moment of inertia.
• The off-diagonal elements are called products of inertia.
The Inertia Tensor

• Equivalently, we separate mass elements to density and volume elements:

\[ \mathbf{I} = \int_{V} \rho(x, y, z) \begin{bmatrix}
  y^2 + z^2 & -xy & -xz \\
  -xy & z^2 + x^2 & -yz \\
  -xz & -yz & x^2 + y^2
\end{bmatrix} dx \, dy \, dz \]

• The diagonal elements: distances to the respective principal axes.
• The non-diagonal elements: products of the perpendicular distances to the respective planes.
Moment of Inertia

- The moment of inertia $I_e$, with respect to a rotation axis $\vec{e}$, measures how much the mass "spreads out":

$$I_e = \int_V (r_e)^2 \, dm$$

- $r_e$: perpendicular distance to axis.
- Through the central rotation origin point.

- Measures ability to resist change in rotational motion.
  - The angular equivalent to mass!
Moment and Tensor

• We have: \((r_e)^2 = |(\hat{e} \times \hat{r})|^2\) for any point \(\hat{q}\).
  • Remember: \(\hat{r}\) is origin \(\rightarrow\) \(\hat{q}\).
  • Then, \(\hat{e} \times \hat{r}\) is closest point on axis \(\rightarrow\) \(\hat{q}\).

• We get:
  
  \[
  \mathbf{I}_e = \int_M |(\hat{e} \times \hat{r})|^2 \, dm = \hat{e}^T \mathbf{I} \hat{e}
  \]

• The **scalar angular momentum** around the axis is then \(L_e = \mathbf{I}_e \omega_e\).
  • \(\omega_e\) is the **angular speed** around \(\hat{e}\).

• Reducible to a planar problem (axis as Z axis).
Change of coordinates

• Suppose we have \( \mathbf{I}, \vec{\omega} \) in the canonical system.
• How do we find them in a rotated system \( R \)?
• \( \vec{\omega}_R = R^T \vec{\omega} \).
  • Just a rotation of the axis!
• Insight: \( \mathbf{I} = \int_V \left[ |\vec{r}|^2 \mathbf{I}_{3 \times 3} - \vec{r} \cdot \vec{r}^T \right] dm \)
• As \( \vec{r}_R = R^T \vec{r} \):
  \[
  I_R = |\vec{r}|^2 \mathbf{I}_{3 \times 3} - R^T \vec{r} \cdot \vec{r}^T R = \\
  = R^T (|\vec{r}|^2 \mathbf{I}_{3 \times 3} - \vec{r} \cdot \vec{r}^T) R = R^T \mathbf{I} R.
  \]
• Angular momentum is then:
  \[
  L_R = \mathbf{I}_R \vec{\omega}_R = R^T \mathbf{I} RR^T \vec{\omega} = R^T L
  \]
• Just the rotated vector!
Warning

• We said that angular velocity is the same regardless of the chosen (moving) center in the object axis system.
• But that is the angular velocity around this center.
• It does not mean that it is the same as the angular velocity around the origin of the world!
• Angular velocity appears even without “rotation”.
• Always know who \( \vec{r} \) and \( \vec{v} \) are measured against.
  • More geometrically: who is the axis of rotation.
  • Look at chalkboard.
Moment of Inertia

- For a mass point:
  \[ I = m \cdot r_u^2 \]

- For a collection of mass points:
  \[ I = \sum_i m_i r_i^2 \]

- For a continuous mass distribution on the plane:
  \[ I = \int_M r_u^2 \, dm \]
Inertia of Primitive Shapes

• For primitive shapes, the inertia can be expressed with the parameters of the shape

• Illustration on a solid sphere
  • Calculating inertia by integration of thin discs along one axis (e.g. $z$).
  • Surface equation: $x^2 + y^2 + z^2 = R^2$
Inertia of Primitive Shapes

• Distance to axis of rotation is the radius of the disc at the cross section along $z$: $r^2 = x^2 + y^2 = R^2 - z^2$.

• Summing moments of inertia of small cylinders of inertia $I_z = \frac{r^2 m}{2}$ along the $z$-axis:

$$dI_z = \frac{1}{2} r^2 dm = \frac{1}{2} r^2 \rho dV = \frac{1}{2} r^2 \rho \pi r^2 dz$$

• We get:

$$I_z = \frac{1}{2} \rho \pi \int_{-R}^{R} r^4 dz = \frac{1}{2} \rho \pi \int_{-R}^{R} (R^2 - z^2)^2 dz = \frac{1}{2} \rho \pi [R^4 z - 2R^2 z^3/3 + z^5/5]_{-R}^{R} = \rho \pi (1 - 2/3 + 1/5)R^5.$$

• As $m = \rho (4/3) \pi R^3$, we finally obtain: $I_z = \frac{2}{5} m R^2$. 
Inertia of Primitive Shapes

- Solid sphere, radius $r$ and mass $m$:

$$I = \begin{bmatrix} \frac{2}{5}mr^2 & 0 & 0 \\ 0 & \frac{2}{5}mr^2 & 0 \\ 0 & 0 & \frac{2}{5}mr^2 \end{bmatrix}$$

- Hollow sphere, radius $r$ and mass $m$:

$$I = \begin{bmatrix} \frac{2}{3}mr^2 & 0 & 0 \\ 0 & \frac{2}{3}mr^2 & 0 \\ 0 & 0 & \frac{2}{3}mr^2 \end{bmatrix}$$
Inertia of Primitive Shapes

• Solid ellipsoid, semi-axes $a$, $b$, $c$ and mass $m$:

$$I = \begin{bmatrix}
\frac{1}{5}m(b^2 + c^2) & 0 & 0 \\
0 & \frac{1}{5}m(a^2 + c^2) & 0 \\
0 & 0 & \frac{1}{5}m(a^2 + b^2)
\end{bmatrix}$$

• Solid box, width $w$, height $h$, depth $d$ and mass $m$:

$$I = \begin{bmatrix}
\frac{1}{12}m(h^2 + d^2) & 0 & 0 \\
0 & \frac{1}{12}m(w^2 + d^2) & 0 \\
0 & 0 & \frac{1}{12}m(w^2 + h^2)
\end{bmatrix}$$
Inertia of Primitive Shapes

• Solid cylinder, radius $r$, height $h$ and mass $m$: 

\[
I = \begin{bmatrix}
\frac{1}{12}m(3r^2 + h^2) & 0 & 0 \\
0 & \frac{1}{12}m(3r^2 + h^2) & 0 \\
0 & 0 & \frac{1}{2}mr^2
\end{bmatrix}
\]

• Hollow cylinder, radius $r$, height $h$ and mass $m$: 

\[
I = \begin{bmatrix}
\frac{1}{12}m(6r^2 + h^2) & 0 & 0 \\
0 & \frac{1}{12}m(6r^2 + h^2) & 0 \\
0 & 0 & mr^2
\end{bmatrix}
\]
Parallel-Axis Theorem

- We can compute the moment of inertia around an axis $\vec{z}$ going through the COM.
- How to efficiently calculate it for any parallel axis $\vec{z}'$?
- parallel axis theorem:

$$I_{z'} = I_z + md^2$$

- $d$ is the distance between the axes.
- **Summary:** it is easy to find the moment of inertia for every axis and center from the “built-in” one.
Parallel-Axis Theorem

• More generally, for point displacements: $(d_x, d_y, d_z)$

\[
\begin{align*}
I_{xx} &= \int (y^2 + z^2)\,dm + md_x^2 \\
I_{yy} &= \int (z^2 + x^2)\,dm + md_y^2 \\
I_{zz} &= \int (x^2 + y^2)\,dm + md_z^2 \\
I_{xy} &= \int (xy)\,dm + md_x d_y \\
I_{xz} &= \int (xz)\,dm + md_x d_z \\
I_{yz} &= \int (yz)\,dm + md_y d_z
\end{align*}
\]
Torque

• A force \( \vec{F} \) applied at a distance \( r \) from the origin.

• Tangential part causes tangential acceleration:
  \[ \vec{F}_\perp = m \cdot \vec{a}_\perp \]

• The torque \( \vec{\tau} \) is defined as:
  \[ \vec{\tau} = \vec{r} \times \vec{F} \]

• So we get
  \[ \tau = m \cdot (r \cdot \alpha) \cdot r = mr^2 \alpha. \]
  • unit is \( N \cdot m \)
  • Induces the rotation of the system.
Newton’s Second Law

- The law $\vec{F} = m \cdot \vec{a}$ has an equivalent formulation with the inertia tensor and torque:

  \[ \vec{\tau} = \mathbf{I} \vec{\alpha} \]

- Force $\Leftrightarrow$ linear acceleration
- Torque $\Leftrightarrow$ angular acceleration
Torque and Angular Momentum

• **Reminder:** linear: $\vec{F} = \frac{d\vec{p}}{dt}$ ($\vec{p}$: linear momentum).

• Similarly with **torque and angular momentum**:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F} = 0 + \vec{\tau}$$

• **Momentum-torque relation:**

$$\frac{d\vec{L}}{dt} = \frac{d(I\vec{\omega})}{dt} = I \frac{d\vec{\omega}}{dt} = I\vec{\alpha} = \vec{\tau}$$

• **Force** $\Leftrightarrow$ derivative of linear momentum.

• **Torque** $\Leftrightarrow$ derivative of angular momentum.
Rotational Kinetic Energy

• Translating energy formulas to rotational motion.

• The **rotational kinetic energy** is defined as:

\[
E_{Kr} = \frac{1}{2} \bar{\omega}^T \cdot I \cdot \bar{\omega}
\]
Conservation of Mechanical Energy

• Adding rotational kinetic energy

\[ E_{Kt}(t + \Delta t) + E_P(t + \Delta t) + E_{Kr}(t + \Delta t) \]
\[ = E_{Kt}(t) + E_P(t) + E_{Kr}(t) + E_O \]

• \( E_{Kt} \) is the translational kinetic energy.
• \( E_P \) is the potential energy.
• \( E_{Kr} \) is the rotational kinetic energy.
• \( E_O \) the “lost” energies (surface friction, air resistance etc.).
Torque Impulse

- We may apply off-center forces for a very short amount of time.
  - Or as a collision.

- torque impulse $\rightarrow$ instantaneous change in angular momentum, i.e. in angular velocity:

$$\tau \Delta t = \Delta L$$
Rigid Body Forces

• A force can be applied anywhere on the object, producing also a rotational motion.
• Q: how come two non-rotating objects can cause a rotation?
Complex Objects

• When an object consists of multiple primitive shapes:
  • Calculate the individual moment inertia of each shape around a the prescribed axis in the same coordinates system, and their individual origins.
  • Use parallel axis theorem to transform to inertia to unified object coordinates.
  • Add the moments together.