Lecture VII: Soft-Body Physics
Soft Bodies

• Realistic objects are not purely rigid.
  • Good approximation for “hard” ones.
  • ...approximation breaks when objects break, or deform.

• Generalization: soft (deformable) bodies
  • Deformed by force: car body, punched or shot at.
  • Prone to stress: piece of cloth, flag, paper sheet.
  • Not solid: snow, mud, lava, liquid.

http://www.games73.com/media.games73.com/files/2012/05/Crysis-3-soft-physics-demo-thumb-610x239.jpg
http://i.huffpost.com/gen/1480563/images/o-DISNEY-facebook.jpg
Grinspun et al. “Discrete Shells”
Elasticity

• Forces may cause object deformation.

• **Elasticity**: the tendency of a body to **return to its original shape** after the forces causing the deformation cease.
  • Rubbers are highly elastic.
  • Metal rods are much less.


http://www.ibmbigdatahub.com/sites/default/files/elasticity_blog.jpg
Continuum Mechanics

• A deformable object is defined by rest shape and material parameters.

• Deformation map: $f(\mathbf{x})$ of every material point $\mathbf{x}$.

• $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$. $d$: dimension (mostly $d = 2, 3$).

• Relative displacement field: $f(\mathbf{x}) = \mathbf{x} + u(\mathbf{x})$. 
Local Deformation

• Taylor series:

\[ f(\vec{x} + \Delta \vec{x}) \approx f(\vec{x}) + J_f \Delta \vec{x} \]

• 1\textsuperscript{st}-order linear approximation.

• As \( f(\vec{x}) = \vec{x} + u(\vec{x}) \), we get:

\[ \vec{x} + \Delta \vec{x} + u(\vec{x} + \Delta \vec{x}) \approx \vec{x} + u(\vec{x}) + J_f \Delta \vec{x} \]

\[ u(\vec{x} + \Delta \vec{x}) \approx u(\vec{x}) + \left( J_f - I_{d \times d} \right) \Delta \vec{x} \]

• The Jacobians:

\[ J_f = \left( \frac{\partial f}{\partial x} \right), J_u = \left( \frac{\partial u}{\partial x} \right) = J_f - I_{d \times d} \]
Stretch and Compression

• How much an object locally stretches or compresses in each direction.

• New length:

\[ |\Delta f|^2 = |f(\vec{x} + \Delta \vec{x}) - f(\vec{x})|^2 \approx |J_f \Delta \vec{x}|^2 \]
\[ = \Delta \vec{x}^T (J_f^T J_f) \Delta \vec{x} \]

• Stretch: relative change in length:

\[ \frac{|\Delta f|^2}{|\Delta x|^2} \approx \frac{\Delta \vec{x}^T (J_f^T J_f) \Delta \vec{x}}{\Delta \vec{x}^T \Delta \vec{x}} \]

Rigid-Body Deformation

• Transformation:
  \[ f(\vec{x}) = R\vec{x} + T \]
  • \( R \): rotation (constant)
  • \( T \): translation.

• \( J_f = R \), and then \( J_f^T J_f = I \).
  • No stretch!

Cauchy-Green Deformation Tensor

• \((J_f^T J_f)_{d \times d}\) is the (right) Cauchy-Green tensor.

• Interpretation: for direction \(\hat{a}\), we get:
  \[|\nabla_{\hat{a}}(f(x))|^2 \approx \hat{a}^T \cdot (J_f^T J_f) \cdot \hat{a}\]

• In words: the CG tensor measures the ratio of change in squared length in a direction.
The Green-Lagrange Strain Tensor

• Measures the deviation from rigidity:

\[ E_{3 \times 3} = \frac{1}{2} (J_f^T J_f - I) \]

• Gives “0” for rotations.
  • Rather than “1” in the CG tensor.

• In deformation field terms \((J_f = J_u - I_{d \times d})\):

\[ E = \frac{1}{2} (J_u^T J_u + J_u + J_u^T) \]
Strain

• The fractional deformation $\epsilon = \Delta L / L$
  • Dimensionless (a ratio).
  • How much a deformation differs from rigidity in a given direction:
    • Negative: compression
    • Zero: rigid
    • Positive: stretch

• In 1D: $\frac{|\Delta f|}{|\Delta x|} = \frac{\Delta L + L}{L} = 1 + \epsilon$
GL Tensor and Strain

- For Strain $\epsilon = \Delta L/L$ in (unit length) direction $\hat{d}$:
  $$\hat{d}^T E \hat{d} = \epsilon + \frac{1}{2} \epsilon^2$$

- Problem: the GL strain tensor is nonlinear if the deformation $f$ (or the deformation field $u$).

- Approximation: the infinitesimal strain tensor:
  $$\epsilon = \frac{1}{2} \left( J_u^T J_u + J_u + J_u^T \right) \approx \frac{1}{2} \left( J_u + J_u^T \right)$$
  Where $\hat{d}^T E \hat{d} \approx \epsilon$. 

$E_{3x3} = \frac{1}{2} (J_f^T J_f - I)$
The Infinitesimal Strain Tensor

• A.K.A. Cauchy’s strain tensor:
\[ \varepsilon = \frac{1}{2} \left( J_u + J_u^T \right) = \frac{1}{2} \left( J_f + J_f^T \right) - I \]

• So that for unit direction \( \vec{d} \):
\[ \varepsilon_d = \hat{d}^T \varepsilon \hat{d} \]

• The Strain tensor is not rotation invariant!
• Good for small deformations \( |J_u| \ll 1 \).
Poisson’s ratio

- Strain in one direction usually causes compression in another.

- **Poisson’s ratio**: the ratio of transversal to axial strain:

\[ \nu = -\frac{d \text{ [transversal strain]}}{d \text{ [axial strain]}} \]

- Equals 0.5 in **perfectly incompressible** material.

- If the force is applied along \( x \):

\[ \nu = -\frac{d\varepsilon_y}{d\varepsilon_x} = -\frac{d\varepsilon_z}{d\varepsilon_x} \]
Poisson’s ratio

• Example of a cube of size $L$.

• Average strain in each direction: $\nu \approx \frac{\Delta L'}{\Delta L}$

  • Approximate, because true for small elements and deformation.
Stress

• **Magnitude** of applied force per area of application.
  - large value $\iff$ force is large or surface area is small

• **Pressure measure** $\sigma$.

• **Unit:** Pascal: $[Pa] = \left[ \frac{N}{m^2} \right]$

• Example: gravity stress on plane:
  $$\sigma = \frac{mg}{\pi r^2}$$
The Linear Stress Tensor

• Measuring stress for each (unit) direction \( \hat{n} \) in an infinitesimal volume element:

\[
\sigma(\hat{n}) = \begin{pmatrix}
\sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\
\sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\
\sigma_{xz} & \sigma_{yz} & \sigma_{zz}
\end{pmatrix} \hat{n} = \sigma \hat{n}
\]

• Note that \( \sigma \hat{n} \) is not necessarily parallel to \( \hat{n} \)!

• \( \sigma \hat{n} = \langle \sigma \hat{n}, \hat{n} \rangle \hat{n} + \tau \)

(normal stress) (shear stress)
Body Material

• The amount of stress to produce a strain is a property of the material.

• **Isotropic materials**: same in all directions.

• **Modulus**: a ratio of stress to strain.
  • Usually in a linear direction, along a planar region or throughout a volume region.
    • Young’s modulus, Shear modulus, Bulk modulus
  • Describing the material reaction to stress.

http://openalea.gforge.inria.fr/doc/vplants/mechanics/doc/_build/html/_images/system_geometry7.png
Young’s Modulus

• Defined as the ratio of linear stress to linear strain:

\[ Y = \frac{\text{linear stress}}{\text{linear strain}} = \frac{F/A}{\Delta L/L} \]

• Example:
Shear modulus

• The ratio of **planar stress** to **planar strain**:

\[ S = \frac{\text{planar stress}}{\text{planar strain}} = \frac{F/A}{\Delta L/L} \]

• Example:
Bulk modulus

• The ratio of \textit{volume stress to volume strain} (inverse of compressibility):

\[ B = \frac{\text{volume stress}}{\text{volume strain}} = \frac{\Delta P}{\Delta V/V} \]

• Example
Linear Elasticity

• **Stress** and **strain** are related by Hooke’s law
  • Remember $F = -k\Delta x$?

• Reshape tensors to vector form:

• $\bar{\sigma} = (\sigma_{xx}, \sigma_{xy}, \cdots, \sigma_{zz})$, and similarly for $\bar{\varepsilon}$.

• Then the **stiffness tensor** $C_{9 \times 9}$ holds:
  $$\bar{\sigma} = C\bar{\varepsilon}$$

https://people.eecs.berkeley.edu/~sequin/CS184/TOPICS/SpringMass/Spring_mass_2D.GIF

Hyperelastic Materials

• Seek to return to their “rest shape”.
  • Have a potential deformation energy

• Spring energy: \( E_S = \frac{1}{2} k \| x - x_0 \|^2 \).

• **Underlying assumption**: deformation energy is not path-dependent!
Linear Elasticity Energy

• One Possibility is: \( E = \frac{1}{2} \int_T \langle \bar{\sigma}, \bar{\epsilon} \rangle dV \).
  
  • Possibilities depend on the type of Stress\Strain tensors to use.
  
  • This one is popular for linear elasticity with FEM.

• We get:

\[
E = \frac{1}{2} \int_T \bar{\epsilon} \mathbf{C} \bar{\epsilon} dV
\]
Dynamic Elastic Materials

• For every point \( q \), The PDE is given by

\[
\rho \frac{d^2 \vec{u}}{dt^2} = \nabla \cdot \sigma + \vec{F}
\]

• \( \rho \): the density of the material.

• \( \nabla \cdot \sigma = (\partial / \partial x, \partial / \partial y, \partial / \partial z) \cdot \sigma \) is the divergence of the stress tensor (modeling internal forces).

• \( F \): external body forces (per point)

• Generalized Newton’s 2\textsuperscript{nd} law!
  • Remember \( F = ma \)?
  • Similar, in elasticity language.

http://www.cims.nyu.edu/cmcl/ComplexFluids/Images/OB-Pump.png