Lecture X: Fluid Physics
Fluid Motion

- Describes object with no fixed topology
  - Air flow
  - Viscuous fluids
  - Smoke, etc.
- Key descriptor: **flow velocity**
  \[ \vec{u} = u(x, t) \]
- Describing the velocity of a “fluid parcel” passing at position \( x \) in time \( t \).
- **Eulerian description**
  - How come?
Flow Velocity

- Vector field describing motion
- Steady field: \( \frac{\partial u}{\partial t} = 0 \)
- Incompressible: \( \nabla \cdot u = 0 \).
- Irrotational (no vortices): \( \nabla \times u = 0 \)

Turbulent with a vortex  
Incompressible, irrotational flow  
Steady field
Incompressibility Assumption

• No fluid is incompressible
  • Otherwise, no soundwaves!
• Fluid is a non-solid “rigid body”.
• Explicit condition:
  \[ \iint_{\partial\Omega} \vec{u} \cdot \hat{n} = 0 \]
• For every (sub) domain \( \Omega \) with boundary \( \partial\Omega \).
• “volume in = volume out”.
Incompressibility

• The divergence theorem:

\[ \iiint_{\partial \Omega} \vec{u} \cdot \hat{n} = \iiint_{\Omega} \nabla \cdot \vec{u} = 0 \]

• Since \( \Omega \) is arbitrary, we have:

\[ \nabla \cdot \vec{u} = 0 \]

everywhere as our condition for incompressibility!
(Incompressible) Navier-Stokes Equations

- Representing the conservation of mass and momentum for an incompressible fluid ($\nabla \cdot \vec{u} = 0$):

\[
\frac{D\vec{u}}{Dt} = \rho \left( \vec{u}_t + \vec{u} \cdot \nabla \vec{u} \right) = \nu \nabla \cdot (\nabla \vec{u}) - \nabla p + \vec{f}
\]

- $p$: pressure field
- $\nu$: kinematic viscosity.
- $f$: body force per density (usually just gravity $\rho g$).
Material Derivative

• A particle with mass $m$ has velocity $\vec{u}$.

• Newton’s 2$\text{nd}$ law:

$$m \frac{D\vec{u}}{Dt} = \vec{F}$$

• What are the forces acting on the particle?

• Simplest component: gravity $\vec{F}_g = m\vec{g}$. 
Pressure

• The particle receives pressure from all directions in a fluid.

• What causes pressure? Incompressibility!
  • Pressure balances flow that creates compression.

• Algebraically:
  • $\nabla \cdot \vec{u} = 0$ is a constraint
  • Pressure is the Lagrange multiplier $\lambda$!

• Suppose particle volume $V$.

• Integrated (scalar!) pressures over particle: $p$.

• Net force enacted by pressure: $\vec{F}_{pre} = -V \nabla p$. 
Viscosity

- Resistance to deformation by shear stress.
- Expressed by coefficient $\nu$:
  \[ \frac{F}{A} = \nu \frac{\partial u}{\partial y} \]
  \[ \frac{A}{F} = \frac{1}{\nu} \frac{\partial u}{\partial y} \]
- Higher $\nu$: more pressure required for shearing!
  - Viscid fluids.
Viscosity

- Another interpretation: how much the flow of a particle different from the flow of its neighbors.

- **Laplacian:** \( \nu \nabla \cdot (\nabla \vec{u}) = \nu \Delta \vec{u} \).
  - \( \nu \): dynamic viscosity.

- \( \nu \) is defined per volume unit, so to get force:

- \( \vec{F}_{vis} = V \cdot [\nu \nabla \cdot (\nabla \vec{u})] \)

- What is the Laplacian of a vector?
  - For each coordinate independently.
Putting it All Together

• The Navier-Stokes equation in particle form:

\[
m \frac{D \mathbf{u}}{Dt} = \mathbf{F}_{vis} + \mathbf{F}_{pre} + \mathbf{F}_{ext} = V \nabla \cdot (\nabla \mathbf{u}) - V \nabla p + m \mathbf{g}
\]

• In the limit (infinitesimal particles): \( \frac{m}{V} \to \rho \).

• dividing by \( V \) we get:

\[
\rho \frac{D \mathbf{u}}{Dt} = \nu \nabla \cdot (\nabla \mathbf{u}) - \nabla p + \rho \mathbf{g}
\]

• What is the material derivative \( \frac{D \mathbf{u}}{Dt} \)?
Material Derivative

• The connecting thread between Lagrangian and Eulerian approaches.

• Reminder:
  • Lagrangian: tracking elements as they move (particles, mesh vertices).
  • Eulerian: tracking movement in a fixed point in space (temperature on grid, amount of material in a cell).

• Lagrangian description:
  • Suppose a set of particles, each with quantity $q$.
  • The particle is moving with velocity $\vec{u}(t) = \frac{d\vec{x}(t)}{dt}$

• Eulerian description:
  • $q(t, \vec{x})$: the quantity of the particle that passes through fixed point $\vec{x}$ at time $t$.

• How to tie both narratives?
Material Derivative

• Material (or total) derivative:

\[
\frac{Dq}{Dt} = \frac{d}{dt} q(t, \hat{x}(t))
\]

• How much \( q \) changes for the moving particle.

\[
\frac{\partial}{\partial t} q(t, \hat{x})
\]

• How much \( q \) changes at a fixed point \( \hat{x} \).
Material Derivative

• The chain rule:

\[
\frac{Dq}{Dt} = \frac{d}{dt} q(t, \mathbf{x}(t)) = \frac{\partial q}{\partial t} + \nabla q \cdot \frac{d\mathbf{x}}{dt} = \frac{\partial q}{\partial t} + \nabla q \cdot \mathbf{u}
\]

• \( \frac{\partial q}{\partial t} \): How much \( q \) changes at the fixed point \( \mathbf{x} \).

• \( \nabla q \cdot \mathbf{u} \): how much of the change is due to advection (transfer of material).
  • Also: convection, transport.

• When \( q \) doesn’t change for a particle, we get the advection equation:

\[
\frac{\partial q}{\partial t} + \nabla q \cdot \mathbf{u} = 0
\]
Material Derivative

• The change in the velocity of the fluid parcel passing at position $x$ in time $t$.

$$\frac{D\vec{u}}{dt} = \vec{u}_t + \vec{u} \cdot \nabla \vec{u}$$

• $\vec{u}_t$: unsteady acceleration.
  • How much velocity changes in fixed $x$ over time.

• $\vec{u} \cdot \nabla \vec{u}$: convective acceleration.
  • How much velocity changes due to movement along trajectory.

1 http://www.continuummechanics.org/images/continuity/converging_nozzle.png
Inviscid Flow

• We often drop viscosity altogether.

• Getting the Euler equations:

\[ m \frac{D\vec{u}}{Dt} = \vec{F}_{pre} + \vec{F}_{ext} = -\nabla p + mg \]

• “numerical dissipation”: errors that mimic physical viscosity enough for the visual effet.
  • Numerical dissipation often >> true physical viscosity!
Boundary Conditions

• **Solid walls**: no velocity through:
  \[ \mathbf{u} \cdot \mathbf{n} = 0 \]
  
  • What about **pressure**?

• For viscid fluid, there is also tangential “stick”.
  • “no-slip” conditions: \( \mathbf{u} = 0 \)
Water Surface

• Actually a meeting between two fluids (water and air).

• We can assume $p_{AIR} = 0$

• Model: water tries to minimize surface area.

https://abm-website-assets.s3.amazonaws.com/laboratoryequipment.com/s3fs-public/legacyimages/061813_1oW.jpg
Water Surface

• Pressure jump: $[p] = \gamma \kappa$

• $\gamma$: surface tension coefficient

• $\kappa$: surface mean curvature.

• Mean curvature: measure of area gradient.

$$2\kappa \hat{n} = \lim_{A \to 0} \frac{\nabla A}{A}$$