1. Rigid-body Motion

1.1. The predictable Spider-Man. Spider-Man, weighing $m = 75$ Kg, is on the top of a building at coordinate $\vec{p} = (0, 50)$ (neglect width of building). He needs to get to the top of another building at $\vec{q} = (70, 80)$, with nothing in between. An inexplicable floating platform, covering the line $y = 120$, would assist him in this. He does that by shooting a string to attach to platform, and swings on it to get to the other building.

(1) What should be the coordinates of the point where the string attaches to the platform?

(2) At what velocity $v_p$, orthogonal to the string, does Spider-Man have to jump to be able to reach the building at $q$? Neglect any friction.

solution

(1) Let us denote the attachment point as $\vec{c}$. Then, the condition is that $|\vec{c} - \vec{p}|^2 = |\vec{c} - \vec{q}|^2$, under the condition $c_y = y$. We get:

$$(c_x - p_x)^2 + (c_y - p_y)^2 = (c_x - q_x)^2 + (c_y - q_y)^2 \Rightarrow$$

$$2(q_x - p_x)c_x + p_x^2 - q_x^2 = (y - q_y)^2 - (y - p_y)^2 \Rightarrow$$

$$c_x = \frac{(y - q_y)^2 - (y - p_y)^2 + q_x^2 - p_x^2}{2(q_x - p_x)} = \frac{40^2 - 70^2 + 70^2 - 0^2}{2 \cdot 70} \approx 11,429$$

(2) The string doesn’t even matter here; the minimal velocity $v_p$, under preservation of energy, should be at least enough to counter for the potential energy from the height difference $h = q_y - p_y = 30m$. Therefore:

$$mgh = \frac{1}{2}mv_p^2 \Rightarrow$$

$$v_p = \sqrt{2gh} \approx 24.26 \frac{m}{sec}$$

1.2. Elevator to the stars. A person of $m = 80$ Kg is going up in an open-roof elevator (neglect breathing and other atmospheric issues) from the surface of the earth in an upwards total acceleration of $a = 50 \frac{m}{sec^2}$.

(1) The person stands on a generic supermarket human weighting scale installed in the elevator for some reason. What does the scale show?
(2) The elevator halts abruptly at a height \( h \) to allow the person to comfortably be ejected to retirement in perpetual motion away from earth to outer space. What is \( h \) to allow the person to escape earth’s gravity? assume it’s not high enough to modify \( g \).

**solution**

(1) The supermarket weight, designed to stay on a flat and static floor, computes one’s mass as \( \frac{W}{g} \), where \( W \) is the force (weight) that the person presses unto the scale. To create an acceleration of \( a \) on the person, the elevator must also negate gravity. Thus, the total upwards force is \( W = m(a + g) = 4784 \text{N} \). Therefore, the person produces an equal and opposite weight upon the scale, that then show a false mass of \( \frac{4784}{9.81} \approx 487.66 \text{Kg} \).

(2) This is called escape velocity, and happens when the gravitational potential energy \( mgh \) equals the kinetic velocity. Assuming unmodified \( g \), it is about \( v \approx 11186 \frac{m}{\text{sec}} \) on earth. Then, as we have \( v = \sqrt{2ha} \), we get \( h = \frac{v^2}{2a} \approx 1251.27 \text{Km} \) (note unit).

2. **Calculus**

2.1. **Maps and Transformations.**

2.1.1. *The lost matrix.* Suppose there is a metric \( M(u, v) \) that is known to be positive semidefinite for vectors \( u \in \mathbb{R}^d \). Unfortunately, we lost the actual matrix representing \( M \); nevertheless, we still have the black-box function \( M(u, v) \). What is the minimal amount of function calls we need, as a pre-process, to have enough information, so that later we can measure the length of every possible vector \( u \) as \( |u|_M = M(u, u) \) without more function calls? explain. What is the conclusion on the diagonal values of the missing matrix?

**Solution** Feeding the canonical vectors \( e_i = (0, \ldots, 1, \ldots, 0) \), with “1” in the \( i \)th place, \( d \) times to \( M \) would produce the length of every vector in the canonical axis system. This is enough to measure the length of every vector \( U \in \mathbb{R}^d \) because of the linearity of metrics: for the canonical basis \( e_1, \ldots, e_d \), where every vector \( u = \sum \alpha_i e_i \), we get \( |u|_M = \sum \alpha_i^2 M(e_i, e_i) \). In fact, \( M(e_i, e_i) \) are exactly the diagonal values of the lost matrix \( M \), and that means that PSD matrices have positive (and real) diagonal values. **Note:** the entire matrix did not need to be reconstructed.

2.1.2. *Cross-products as rotations.* Show how a cross product \( \vec{a} \times \vec{b} \) can be written as \( Ab \) where \( A \) is a \( 3 \times 3 \) matrix. Further explicitly show that when \( \vec{b} \) is in a plane, and \( \vec{a} = \hat{n} \) is the normal to the plane, this operator matrix amounts to a 90° rotation in the plane.

**Solution** This is a linear operator on \( b \), and therefore always possible to represent using a matrix. For \( a = (a_x, a_y, a_z) \) we have that:

\[
\begin{pmatrix}
 0 & -a_z & a_y \\
 a_z & 0 & -a_x \\
-a_y & a_x & 0
\end{pmatrix} b
\]
$c = \hat{n} \times b$ is always orthogonal to both, and therefore in the plane with $b$. Without loss of generality, we assume coordinates where the plane is the $xy$ plane where $b_z = 0$, and $\hat{n} = (0, 0, 1)$. The matrix then becomes:

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} b$$

Which can be reduced to:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \end{pmatrix}$$

which is a $90^\circ$ rotation.

2.2. **Vector calculus.** We know the Laplace operator $\Delta \phi = \nabla \cdot (\nabla \phi)$ (the divergence of the gradient of scalar field $\phi$) for scalar functions. We call $\phi$ harmonic if $\Delta \phi = 0$. We call a vector field $\vec{v}$ harmonic if $\nabla \times \vec{v} = 0$ (curl free, irrotational) and $\nabla \cdot \vec{v} = 0$ (divergence free).

1. Show that the gradient of a harmonic scalar function is a harmonic vector field. In which kind of domains is the converse (harmonic field $\rightarrow$ gradient of harmonic function) true?

2. Assume we are in a two-dimensional domain. For a harmonic field $H$, we create a new field $H^\perp$ by rotating each vector in $H$ by $90^\circ$. Show that $H^\perp$ is also harmonic (neglect boundary issues). **Hint:** research for curl-divergence-cross product-dot product identities.

**Solution**

1. By definition we have that $\nabla \times \nabla \phi = 0$, and from the harmonic condition we have that $\nabla \cdot \nabla \phi = 0$. As such, $\vec{v} = \nabla \phi$ is by definition harmonic. The converse (div-free and curl-free $\Rightarrow$ gradient of harmonic scalar function) is again only true only for simply-connected domains.

2. From the previous question we know that, given normal $n$ to that plane, that every vector $h \in H$ gives rises to vector $\hat{n} \times h \in H^\perp$ in the orthogonal field. We then calculate:

$$\nabla \cdot (\hat{n} \times a) = (\nabla \times \hat{n}) \cdot h - \hat{n} \cdot (\nabla \times a) = 0 \cdot h - \hat{n} \cdot 0 = 0.$$  
$$\nabla \times (\hat{n} \times a) = \hat{n} (\nabla \cdot a) - a (\nabla \cdot \hat{n}) + (a \cdot \nabla) \hat{n} - (\hat{n} \cdot \nabla) a = 0$$

We get these equalities as $a$ is harmonic, and as $\hat{n}$ is constant. In fact, it would be enough for $\hat{n}$ to be merely harmonic to have this property as well!

3. **Collisions**

A silver bullet of mass $m_b = 20g$ is shot (horizontally; neglect any vertical gain) with speed $u_b = 2000 \frac{m}{sec}$, hitting a werewolf of mass $m_w = 150kg$ at rest. The bullet “embeds”
in the werewolf and drags it back on the ground. All the kinetic energy that is lost in the
collision is converted into magic energy (measured in J).

(1) What is the speed of the werewolf after the collision?
(2) How much magic energy is generated?
(3) Next suppose that the forest ground has a kinetic friction coefficient of \( \mu = 0.2 \). How far off would the werewolf be dragged from its initial position?

Solution.
(1) The momentum of the system is from the bullet, and is \( m_b u_b \). After the impact, the
momentum is preserved: \( (m_b + m_w)u_w = m_b u_b \) giving \( u_w = \frac{m_b}{m_b + m_w} u_b \approx 0.266 \text{ m sec} \).
(2) The energy difference is \( \frac{1}{2} (m_b v_b^2 - (m_b + m_w) v_w^2) \approx 40000 - 5.31 \) = 39994.69 J.
(3) The force in the other direction equals \( \mu (m_b + m_w) g = 294.33 J \), giving deceleration
of \( 1.962 \frac{m}{\text{sec}^2} \). We then get the small distance of \( d = \frac{v_w^2}{2a} \approx 1.8 \text{ cm} \). Note cm units—
life is not a Tarantino movie! (even life with werewolves)