Lecture VIII: Soft-Body Physics
Soft Bodies

• Realistic objects are not purely rigid.
  • Good approximation for “hard” ones.
  • ...approximation breaks when objects break, or deform.

• Generalization: soft (deformable) bodies
  • Deformed by force: car body, punched or shot at.
  • Prone to stress: piece of cloth, flag, paper sheet.
  • Not solid: snow, mud, lava, liquid.

http://www.games73.com/media.games73.com/files/2012/05/Crysis-3-soft-physics-demo-thumb-610x239.jpg

http://i.huffpost.com/gen/1480563/images/o-DISNEY-facebook.jpg

Grinspun et al. “Discrete Shells”
Elasticity

- Forces may cause object deformation.

- **Elasticity**: the tendency of a body to return to its original shape after the forces causing the deformation cease.
  - Rubbers are highly elastic.
  - Metal rods are much less.


http://www.ibmbigdatahub.com/sites/default/files/elasticity_blog.jpg
A deformable object is defined by rest shape and material parameters.

Deformation map: $f(\mathbf{x})$ of every material point $\mathbf{x}$.

$f: \mathbb{R}^d \rightarrow \mathbb{R}^d$. $d$ : dimension (mostly $d = 2,3$).

Relative displacement field: $f(\mathbf{x}) = \mathbf{x} + u(\mathbf{x})$. 
Local Deformation

• Taylor series:
  \[ f(\tilde{x} + \Delta \tilde{x}) \approx f(\tilde{x}) + J_f \Delta \tilde{x} \]

• 1st-order linear approximation.

• As \( f(\tilde{x}) = \tilde{x} + u(\tilde{x}) \), we get:
  \[ \tilde{x} + \Delta \tilde{x} + u(\tilde{x} + \Delta \tilde{x}) \approx \tilde{x} + u(\tilde{x}) + J_f \Delta \tilde{x} \Rightarrow u(\tilde{x} + \Delta \tilde{x}) \approx u(\tilde{x}) + (J_f - I_{d \times d}) \Delta \tilde{x} \]

• The Jacobians: \( J_f = \left( \frac{\partial f}{\partial x} \right), J_u = \left( \frac{\partial u}{\partial x} \right) = J_f - I_{d \times d} \).
Stretch and Compression

• How much an object locally stretches or compresses in each direction.

• New length:

\[
|\Delta f|^2 = |f(\vec{x} + \Delta \vec{x}) - f(\vec{x})|^2 \approx |J_f \Delta \vec{x}|^2
\]

\[
= \Delta \vec{x}^T \ast (J_f^T J_f) \ast \Delta \vec{x}
\]

• \((J_f^T J_f)_{d \times d}\) is the (right) Cauchy-Green tensor.

• Stretch: relative change in length:

\[
\frac{|\Delta f|^2}{|\Delta p|^2} \approx \frac{\Delta \vec{x}^T \ast (J_f^T J_f) \ast \Delta \vec{x}}{\Delta \vec{x}^T \ast \Delta \vec{x}}
\]
Rigid-Body Deformation

• Transformation:

\[ f(\hat{x}) = R\hat{x} + T \]

• \( R \): rotation (constant)
• \( T \): translation.

• \( J_f = R \), and then \( J_f^T J_f = I \).
  • No stretch!

Strain

• The fractional deformation $\epsilon = \Delta L / L$
  • Dimensionless (a ratio).
  • How much a deformation differs from being rigid:
    • Negative: compression
    • Zero: rigid
    • Positive: stretch

• In our previous notation: $\frac{|\Delta f|}{|\Delta p|} = \frac{\Delta L + L}{L} = 1 + \epsilon$
The Green-Lagrange Strain Tensor

• Measures the deviation from rigidity:

\[ \mathbf{E}_{3\times3} = \frac{1}{2} (J_f^T J_f - I) \]

• In deformation field terms \((J_f = J_u - I_{d\times d})\):

\[ \mathbf{E} = \frac{1}{2} (J_u^T J_u + J_u + J_u^T) \]

• For Strain \(\epsilon = \Delta L/L\) in (unit length) direction \(\hat{n}\):

\[ \hat{n}^T \mathbf{E} \hat{n} = \epsilon + \frac{1}{2} \epsilon^2 \]
Infinitesimal Strain Tensor

• For small shape changes:

\[ \varepsilon = \frac{1}{2} \left( J_u^T J_u + J_u + J_u^T \right) \approx \frac{1}{2} \left( J_u + J_u^T \right) \]

\[ = \frac{1}{2} \left( J_f + J_f^T \right) - I \]

• Not rotation invariant anymore.
• Approximate.
• But linear.
Poisson’s ratio

• Strain in one direction causes compression in another.

• Poisson’s ratio: the ratio of transversal to axial strain:

\[ \nu = -\frac{d [\text{transversal strain}]}{d [\text{axial strain}]} \]

• Equals 0.5 in perfectly incompressible material.

• If the force is applied along \( x \):

\[ \nu = -\frac{d \epsilon_y}{d \epsilon_x} = -\frac{d \epsilon_z}{d \epsilon_x} \]
Poisson’s ratio

- Example of a cube of size $L$.
- Average strain in each direction: $\nu \approx \frac{\Delta L'}{\Delta L}$
  - Approximate, because true for small elements and deformation.
Stress

• **Magnitude** of applied force per **area of application**.
  - large value $\iff$ force is large or surface area is small

• **Pressure measure** $\sigma$.

• **Unit**: *Pascal*: $Pa = N/m^2$

• **Example**: gravity stress on plane:
  \[
  \sigma = \frac{mg}{\pi r^2}
  \]
The Linear Stress Tensor

- Measuring stress for each (unit) direction \( \vec{n} \) in an infinitesimal volume element:

\[
\sigma_n = \begin{pmatrix}
\sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\
\sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\
\sigma_{xz} & \sigma_{yz} & \sigma_{zz}
\end{pmatrix} \vec{n} = T \vec{n}
\]

- Note that \( T \vec{n} \) is not necessarily parallel to \( \vec{n} \)!

\[
T \vec{n} = \sigma_n + \tau
\]

- \( \sigma_n \): outward/inward normal stress.
- \( \tau \): shear stress.
Body Material

• The amount of stress to produce a strain is a property of the material.

• **Isotropic materials**: same in all directions.

• **Modulus**: a ratio of stress to strain.
  • Usually in a linear direction, along a planar region or throughout a volume region.
    • Young’s modulus, Shear modulus, Bulk modulus
  • Describing the material reaction to stress.
Young’s Modulus

• Defined as the ratio of linear stress to linear strain:

\[ Y = \frac{\text{linear stress}}{\text{linear strain}} = \frac{F}{A} \frac{\Delta L}{L} \]

• Example:
Shear modulus

• The ratio of **planar stress to planar strain**:

\[ S = \frac{\text{planar stress}}{\text{planar strain}} = \frac{F/A}{\Delta L/L} \]

• Example:
Bulk modulus

• The ratio of volume stress to volume strain (inverse of compressibility):

\[ B = \frac{\text{volume stress}}{\text{volume strain}} = \frac{\Delta P}{\Delta V/V} \]

• Example
Linear Elasticity

• **Stress** and **strain** are related by Hooke’s law
  • Remember $F = -k\Delta x$?

• Reshape tensors to vector form:
  • $\bar{\sigma} = (\sigma_{xx}, \sigma_{xy}, \cdots, \sigma_{zz})$, and similarly for $\bar{\epsilon}$.
  • Then the **stiffness tensor** $\mathbf{C}_{9 \times 9}$ holds:
    $$\bar{\sigma} = \mathbf{C}\bar{\epsilon}$$

https://people.eecs.berkeley.edu/~sequin/CS184/TOPCS/SpringMass/Spring_mass_2D.GIF

Hyperelastic Materials

• Seek to return to their “rest shape”.
  • Have a potential deformation energy

• Recall: spring energy: \( E_S = \frac{1}{2} k \| x - x_0 \|^2 \)

• **Underlying assumption**: deformation energy is not path-dependent!
Linear Elasticity Energy

• One Possibility is: \( E = \frac{1}{2} \int_T \langle \bar{\sigma}, \bar{\epsilon} \rangle dV. \)
  
  • Possibilities depend on the type of Stress\Strain tensors to use.
  
  • This one is popular for linear elasticity with FEM.

• We get:

\[
E = \frac{1}{2} \int_T \bar{\epsilon} \mathbf{C} \bar{\epsilon} dV
\]
Dynamic Elastic Materials

• For every point $q$, The PDE is given by

$$\rho \frac{d^2 \vec{u}}{dt^2} = \nabla \cdot \sigma + \vec{F}$$

• $\rho$: the density of the material.
• $a$: acceleration of point $q$.
• $\nabla \cdot \sigma = (\partial / \partial x, \partial / \partial y, \partial / \partial z) \star \sigma$ is the divergence of the stress tensor (modeling internal forces).
• $F$: external body forces (per density)

• Generalized Newton’s 2nd law!
  • Remember $F = ma$?
  • Similar, in elasticity language.
Fluid Motion

• Describes object with no fixed topology
  • Air flow
  • Viscous fluids
  • Smoke, etc.

• Key descriptor: flow velocity
  \[ \vec{u} = u(x, t) \]

• Describing the velocity of a “fluid parcel” passing at position \( x \) in time \( t \).

• Eulerian description
  • How come?

http://cfd.solvcon.net/old/research/cylinder2.gif
Flow Velocity

- Vector field describing motion
- Steady field: $\frac{du}{dt} = 0$
- Incompressible: $\nabla \cdot u = 0$.
- Irrotational (no vortices): $\nabla \times u = 0$

Turbulent with a vortex
Incompressible, irrotational flow
Steady field
Material Derivative

• The change in the velocity of the fluid parcel passing at position $x$ in time $t$.

$$\frac{D u}{d t} = u_t + u \cdot \nabla u$$

• $u_t$: unsteady acceleration.
  • How much velocity changes in $x$ over time.

• $u \cdot \nabla u$: convective acceleration.
  • How much velocity changes due to movement along trajectory.
Viscosity

• Resistance to deformation by shear stress.
• Expressed by coefficient $\nu$:
  \[
  \frac{F}{A} = \nu \frac{\partial u}{\partial y}
  \]
• Higher $\nu$: more pressure required for shearing!
  • Viscid fluids.
Navier-Stokes Equations

- Representing the conservation of mass and momentum for an incompressible fluid ($\nabla \cdot u = 0$):

\[
\rho \left( u_t + u \cdot \nabla u \right) = \nabla \cdot (\nu \nabla u) - \nabla p + f
\]

- $p$: pressure field
- $\nu$: kinematic viscosity.
- $f$: body force per density (usually just gravity $\rho g$).