Soft-Body Physics
Soft Bodies

• Realistic objects are not purely rigid.
  • Good approximation for “hard” ones.
  • …approximation breaks when objects break, or deform.

• Generalization: soft (deformable) bodies
  • Deformed by force: car body, punched or shot at.
  • Prone to stress: piece of cloth, flag, paper sheet.
  • Not solid: snow, mud, lava, liquid.
Elasticity

• Forces may cause object deformation.

• **Elasticity**: the tendency of a body to return to its original shape after the forces causing the deformation cease.
  - Rubbers are highly elastic
  - Metal rods are much less.
Stress

• The **magnitude** within an object of an applied force, **divided by the area** of the application
  • large value when the force is large or when the surface is small

• It is a **pressure measure** $\sigma$, with a unit denoted as **Pascal**: $Pa = N/m^2$

• Example: the stress on the plane is:
  $$\sigma = mg/(\pi r^2)$$
Strain

• The fractional deformation $\epsilon$ caused by a stress
  • Dimensionless (a ratio).
  • How much a deformation differs from being rigid:
    • Negative: compression
    • Zero: rigid
    • Positive: stretch

• Example
  • the strain on the rod is $\epsilon = \Delta L / L$
Body Material

• The amount of stress to produce a strain is a property of the material.

• **Modulus**: a ratio of **stress** to **strain**.
  
  • Usually in a linear direction, along a planar region or throughout a volume region.
    
    • Young’s modulus, Shear modulus, Bulk modulus
  
  • Describing the material reaction to stress.
Young’s Modulus

- Defined as the ratio of \textit{linear stress} to \textit{linear strain}:

\[ Y = \frac{\text{linear stress}}{\text{linear strain}} = \frac{F/A}{\Delta L/L} \]

- Example:
Shear modulus

• The ratio of planar stress to planar strain:

\[ S = \frac{\text{planar stress}}{\text{planar strain}} = \frac{F/A}{\Delta L/L} \]

• Example:
Bulk modulus

- The ratio of *volume stress* to *volume strain* (inverse of compressibility):

\[ B = \frac{\text{volume stress}}{\text{volume strain}} = \frac{\Delta P}{\Delta V/V} \]

- Example
Poisson’s ratio

• Strain in one direction causes compression in another.

• **Poisson’s ratio**: the ratio of transverse to axial strain:

\[ \nu = -\frac{d \text{ transverse strain}}{d \text{ axial strain}} \]

• Equals 0.5 in **perfectly incompressible** material.

• If the force is applied along \( x \):

\[ \nu = -\frac{d \varepsilon_y}{d \varepsilon_x} = -\frac{d \varepsilon_z}{d \varepsilon_x} \]
Poisson’s ratio

- Example of a cube of size $L$

\[
\begin{align*}
\text{d} \epsilon_x &= \frac{\text{d}x}{x} \\
\text{d} \epsilon_y &= \frac{\text{d}y}{y} \\
\text{d} \epsilon_z &= \frac{\text{d}z}{z}
\end{align*}
\]

\[-\nu \int_L^{L+\Delta L} \frac{\text{d}x}{x} = \int_L^{L-\Delta' L} \frac{\text{d}y}{y} = \int_L^{L-\Delta' L} \frac{\text{d}z}{z} \Leftrightarrow \]

\[
\left(1 + \frac{\Delta L}{L}\right)^{-\nu} = 1 - \frac{\Delta' L}{L} \Leftrightarrow \nu \approx \frac{\Delta' L}{\Delta L}
\]
Continuum Mechanics

- A deformable object is defined by rest shape and material parameters.
- Deformation map: $\mathbf{q} = f(\mathbf{p})$ of every point $\mathbf{p} = (x, y, z)$.
- Relative displacement field: $f(\mathbf{p}) = \mathbf{p} + u(\mathbf{p})$
The strain and stress are related to the material deformation gradient tensor $J_q$, and so to the displacement field $u$.

The stretch in a unit direction $\nu \rightarrow \nu'$:

$$|d\nu'|^2 = \nu^T (J_q^T J_q) \nu$$

$J_q^T J_q$ is called the (right) Cauchy-Green strain tensor.

Movement is rigid $\iff J_q$ is orthonormal $\iff J_q^T J_q = I \iff$ No strain!
Stress Tensor

• And stress tensor from Hooke’s linear material law
  \[ \sigma = E \times \epsilon \]
  where \( E \) is the elasticity tensor and depends on the Young’s modulus and Poisson’s ratio (and more).
Soft-Body Simulation

• Two common approaches to simulation:
  • **Lagrangian** (particle-based):  
    • A set of moving points carrying material properties.  
    • Object is a connected mesh or cloud of points, suitable for deformable soft bodies.  
    • **Examples**: Finite Element/Difference/Volume methods, Mass-spring system, Coupled particle system, Smoothed particle hydrodynamics.
  • **Eulerian** (grid-based):  
    • A stationary point set where material properties change over time.  
    • boundary of object not explicitly defined, suitable for fluids.
Motion of Dynamic Elastic materials

- For every point $q$, The PDE is given by
  $$\rho \ast \alpha = \nabla \cdot \sigma + F$$

- $\rho$: the **density** of the material.
- $\alpha$: acceleration of point $q$.
- $\nabla \cdot \sigma = (d/dx, d/dy, d/dz) \ast \sigma$ is the **divergence** of the stress tensor (internal forces):
- $F$: other external forces.
Finite Element Method (FEM)

• Used to numerically solve partial differential equations (PDEs).
• Tessellating the volume into a large finite number of disjoint elements (3D volumetric/surface mesh).
• Typical workflow:
  • Estimating deformation field from nodes.
  • Computing local strain and stress tensors
  • The motion equation determined by integrating the stress field over each element.
Finite Differences Method

• If the object is sampled using a regular spatial grid, the PDE can be discretized using finite differences.
  • Pro: easier to implement than FEM.
  • Con: difficult to approximate complex boundaries.

• Semi-implicit integration is used to move forward through time
Boundary Element Method

• The boundary element method simplifies the finite element method from a 3D volume problem to a 2D surface problem.
  • PDE is given for boundary deformation.
  • Only works for homogenoous material.
  • Topological changes more difficult to handle.
Mass-Spring System

• An object consists of point masses connected by a network of massless springs.
• The state of the system: the positions $x_i$ and velocities $v_i$ of the masses $i = 1 \cdots n$.
• The sum force $f_i$ on each mass:
  • External forces (e.g. gravity, friction).
  • Spring connections with the mass’ neighbors.
• The motion equation $f_i = m_i a_i$ is summed up:
  $$M \ast a = f(x, v)$$
  where $M$ is a $3n \times 3n$ diagonal matrix.
Mass-Spring System

- Mass points are initially regularly spaced in a 3D lattice.
- The edges are connected by structural springs.
  - resist longitudinal deformations
- Opposite corner mass points are connected by shear springs.
  - resist shear deformations.
- The rest lengths define the rest shape of the object.
Mass-Spring System

• The force acting on mass point $i$ generated by the spring connecting $i$ and $j$ is

$$f_i = Ks_i(\left|x_{ij}\right| - l_{ij})\frac{x_{ij}}{|x_{ij}|}$$

where $x_{ij}$ is the vector from positions $v_i$ to $v_j$, $K_i$ is the stiffness of the spring and $l_{ij}$ is the rest length.

• To simulate dissipation of energy, a damping force is added:

$$f_i = Kd_i \left( \frac{(v_j - v_i)^T x_{ij}}{x_{ij}^T x_{ij}} \right) x_{ij}$$
Mass-Spring System

• **Pro**: intuitive and simple to implement.
• **Con**: Not accurate and does not necessarily converges to correct solution.
  • depends on the mesh resolution and topology
  • …and the choice of spring constants.
• Can be good enough for games, especially cloth animation
  • For possible strong *stretching* resistance and weak *bending* resistance.
Coupled Particle System

- Particles interact with each other depending on their spatial relationship.
  - these relationships are dynamic, so geometric and topological changes can take place.
- Each particle $p_i$ has a potential energy $E_{Pi}$.
  - The sum of the pairwise potential energies between the particle $p_i$ and the other particles.

\[ E_{Pi} = \sum_{j \neq i} E_{Pij} \]
Coupled Particle System

• The force $f_i$ applied on the particle at position $p_i$ is

$$f_i = -\nabla p_i E_{pi} = -\sum_{j \neq i} \nabla p_i E_{pij}$$

where $\nabla p_i E_{pi} = \left( \frac{dE_{pi}}{dx_i}, \frac{dE_{pi}}{dy_i}, \frac{dE_{pi}}{dz_i} \right)$

• Reducing computational costs by localizing.
  • potential energies weighted according to distance to particle.
Smoothed Particle Hydrodynamics

- The equation for any quantity $A$ at any point $r$ is given by

$$A(r) = \sum_j m_j \frac{A_j}{\rho_j} W(|r - r_j|, h)$$

- $W$ is a smoothing kernel.
  - usually Gaussian function or cubic spline.
- $h$ the smoothing length (max influence distance).
- Example: the density can be calculated as

$$\rho(r) = \sum_j m_j W(|r - r_j|, h)$$

- It is applied to pressure and viscosity forces, while external forces are applied directly to the particles.
Smoothed Particle Hydrodynamics

- **Derivatives of quantities**: by derivatives of $W$.
- Varying the smoothing length $h$ tunes the resolution of a simulation locally.
  - Typically use a large length in low particle density regions and vice versa.
- **Pro**: easy to conserve mass (constant number of particles).
- **Con**: difficult to maintain material incompressibility.
Eulerian Methods

• Typically used to simulate fluids (liquids, smoke, lava, cloud, etc.).

• The scene is represented as a regular voxel grid, and fluid dynamics describes the displacements
  • Applying finite difference formulation on the voxel grid.
  • Velocity is stored on the cell faces.
  • Pressure is stored at the center of the cells.

• Heavily rely on the Navier-Stokes equations of motion for a fluid.
Navier-Stokes Equations

• Representing the conservation of mass and momentum for an incompressible fluid:

\[ \nabla \cdot \mathbf{u} = 0 \]

\[
\rho \left( \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \nabla \cdot (\nu \, \nabla \mathbf{u}) - \nabla p + f
\]

- \( \mathbf{u}_t \) is the time derivative of the fluid velocity (the unknown), \( p \) is the pressure field, \( \nu \) is the kinematic viscosity, \( f \) is the body force per unit mass (usually just gravity \( \rho g \)).
Navier-Stokes Equations

• First $f$ is scaled by the time step and added to the current velocity

• Then the advection term $u \cdot \nabla u$ is solved
  • it governs how a quantity moves with the underlying velocity field (time independent, only spatial effect).
  • it ensures the conservation of momentum.
  • sometimes called convection or transport.
  • solved using a semi-Lagrangian technique.
Then the viscosity term $\nabla \cdot (\nu \nabla u) = \nu \nabla^2 u$ is solved

- it defines how a cell interchanges with its neighbors.
- also referred to as diffusion.
- Viscous fluids can be achieved by applying diffusion to the velocity field.
- it can be solved for example by FD and an explicit formulation:
  - 2-neighbor 1D:
    $$u_i(t) = \nu \Delta t \left( u_{i+1} + u_{i-1} - 2u_i \right)$$
  - 4-neighbor 2D: $$u_{i,j}(t) = \nu \Delta t \left( u_{i+1,j} + u_{i-1,j} + u_{i,j+1} +$$
Navier-Stokes equations

• Finally, the pressure gradient is found so that the final velocity will conserve the volume (i.e. mass for incompressible fluid)
  – sometimes called pressure projection
  – it represents the resistance to compression $-\nabla p$
Navier-Stokes equations

- We make sure the velocity field stays divergence-free with the second equation $\nabla \cdot u = 0$.
- The velocity flux of all faces at each fluid cell is zero (everything that comes in, goes out).
- The equation $u(t + \Delta t) = u(t) - \Delta t \nabla p$ is solved from its combination with $\nabla \cdot u = 0$, giving
  
  $\nabla \cdot u(t + \Delta t) = \nabla \cdot u(t) - \Delta t \nabla \cdot (\nabla p) = 0$
  
  $\iff \Delta t \nabla^2 p = \nabla \cdot u(t)$

  with which we solve for $p$, then plug back in the $u(t + \Delta t)$ equation to calculate the final velocity
Navier-Stokes equations

- Compressible fluids can also conserve mass, but their density must change to do so.
- Pressure on boundary nodes
  - In free surface cells, the fluid can evolve freely ($p = 0$)
    - so that for example a fluid can splash into the air
  - Otherwise (e.g. in contact with a rigid body), the fluid cannot penetrate the body but can flow freely in tangential directions $u_{boundary} \cdot n = u_{body} \cdot n$. 
End of
Soft body physics

Next
Physics engine design and implementation