INTRODUCTION
IMAGE PROCESSING
MATHEMATICAL MORPHOLOGY PART 1

UTRECHT UNIVERSITY
RONALD POPPE
BRIEF RECAP
RECAP

We have looked at:

- Functions that operate on images $g(v)$
- Functions that operate on pixels $g(N)$

We saw kernels that we could enhance

- Contrast
- Edges
- Lines

Now we will look at shapes
OUTLINE

Mathematical morphology
Complement and operator properties

Erosion and dilation
Opening and closing
Geodesic operations and reconstruction
MATHEMATICAL MORPHOLOGY
MATHEMATICAL MORPHOLOGY

Mathematical morphology: the study of shape in an image

Uses a structuring element as a probe:

- Basic operations are erosion and dilation
MATHEMATICAL MORPHOLOGY$^2$

Mathematical morphology works on finite sets $P$ if:

1. We can partially order its elements (with $\leq$), such that:
   \[
   \begin{align*}
   &p_i \leq p_i \\
   &\left( p_i \leq p_j, p_j \leq p_i \right) \rightarrow p_i = p_j \\
   &\left( p_i \leq p_j, p_j \leq p_k \right) \rightarrow p_i \leq p_k
   \end{align*}
   \]

2. There is a *maximum* and a *minimum* for each non-empty subset of $P$
The set of integer numbers is a suitable set $P$:

1. The partial ordering is achieved by the usual “less than or equal to” operator $\leq$

2. The maximum and minimum are defined in their usual sense

Example: $P = \{1, 3, 5\}$

- $1 \leq 3, 3 \leq 5 (1 \leq 5)$
- $\max(P) = \max\{1, 3, 5\} = 5$
- $\min(P) = \min\{1, 3, 5\} = 1$
MATHEMATICAL MORPHOLOGY

The set of all subsets of a superset $S$ is a suitable set $P$:

- The partial ordering is achieved by the subset relation $\subset$
- The maximum is defined by the union $\bigcup$
- The minimum is defined by the intersection $\bigcap$
MATHEMATICAL MORPHOLOGY

Relation between images and sets:

- A binary image is a suitable set $P$
- We can apply morphology to a binary image

A grey value image can be seen as a stack of binary images

- Thresholded at different levels
- $F(c) = \{(x, y) | f(x, y) \geq c\}$
- We can apply morphology to a grey valued image by applying it to all sets in the stack
MATHEMATICAL MORPHOLOGY

For digital images, with a fixed number of possible pixel values:

For two images $f$ and $g$ the $\leq$ relation holds if, for all pixels:

- $f \leq g \iff \forall x : f(x) \leq g(x)$
- “All pixels in $f$ are at least as dark as their counterparts in $g$.”

In sum:

- Morphological operations work on a suitable set $P$
- An image can be regarded as such a set
- So, we can apply morphological operations to images
COMPLEMENT AND OPERATOR
The complement $X^c$ of a set $X$ is defined as all elements not belonging to $X$.

Complementing twice results in the original set: $(X^c)^c = X$.
COMPLEMENT AND OPERATOR²

For images $f$, the complement $f^c$ is defined as $f$ mirrored in a central gray-line value (e.g. 0 or 128)

- Better known as “the inverse”

For an image with range $\{L, \ldots, M\}$:

- $f^c(x,y)=L + M - f(x,y)$ ($L$ usually is 0, $M$ is typically 255)
- Again, complementing twice gives the same image: $(f^c)^c = f$
COMPLEMENT AND OPERATOR

Example: complement of an image

For an image $f$ with range $\{0, \ldots, 8\}$:

- $f^c(x,y) = 8 - f(x,y)$
- Verify that $f(x,y) + f^c(x,y) = 8$

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$f$

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$f^c$

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$f + f^c$
COMPLEMENT AND OPERATOR

Two operators $\varphi_1$ and $\varphi_2$ are dual if:

- $\varphi_1(f^c) = (\varphi_2(f))^c$, for all functions $f$
- Min and max are dual operators

Example:

- For range $\{0, \ldots, 8\}$: $f^c = 8 - f$
- $\max(f^c) = 8$, $\min(f) = 0$
- $(\min(f))^c = 8 - \min(f) = 8$

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Operators $\varphi$ can have different properties:

**Self-dual**: $\varphi(f^c) = (\varphi(f))^c$
- The median filter is self-dual

**Increasing**: $\forall f, g: f \leq g \rightarrow \varphi(f) \leq \varphi(g)$
- It does not alter the order of images
COMPLEMENT AND OPERATOR

Extensive: \( \forall f: f \leq \varphi(f) \)

- The output is always “larger” than the input
- “Making an image lighter”

Anti-extensive: \( \forall f: \varphi(f) \leq f \)

- The reverse of extensive

Idem-potent: \( \forall f: \varphi(\varphi(f)) = \varphi(f) \)

- Applying the operator multiple times has no effect
EROSION AND DILATION
Erosion and Dilation

Morphological operations use (small) structuring elements:

- Usually (approximations of) squares, circles etc.
- Typically binary
- Note: black is “on”/1, white is “off”/0
EROSION AND DILATION²

The dilation $\delta(X)$ of a set (binary image) $X$ by the structuring element $S$ is defined by:

- $\delta(X) = \{x + s | x \in X \land s \in S\}$

In words:

- Place the structuring element $S$ everywhere in the image
- If it hits the set, the center of the element is part of the dilation $\delta$
Example: dilation of $X$ with $S$

- Place the structuring element everywhere in the image
- If it hits the set, the center of the element is part of the dilation $\delta$
EROSION AND DILATION

Dilation: “re-tracing a binary image with a brush with the shape of S”

- Place the center of S on each pixel in X and “color” all pixels under S
EROSION AND DILATION

Erosion is the dual-operator of dilation:

\[ \varepsilon(X) = \{ x \mid \forall s \in S, x + s \in X \} \]

In words:

- Place the structuring element everywhere in the image
- If it is fully contained in the set, the center of the structuring element is part of the erosion
EROSION AND DILATION

Example: erosion of $X$ with $S$

- Place the structuring element everywhere in the image
- If it is fully contained in the set, the center of the structuring element is part of the erosion
EROSION AND DILATION

Example:

Original  Erosion with a square structuring element  Erosion with a larger square structuring element
EROSION AND DILATION

Applying the set definition for all the binary level sets gives us:

\[ \begin{align*}
(\varepsilon(f))(x) &= \min_{s \in S} f(x + s) \\
(\delta(f))(x) &= \max_{s \in S} f(x + s)
\end{align*} \]
**EROSION AND DILATION**

**Exercise:**
- Compute the erosion and dilation of the following image using a 3x3 plus element

![Image of a 3x3 plus element]

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EROSION AND DILATION\textsuperscript{10}
EROSION AND DILATION\textsuperscript{11}
EROSION AND DILATION

Properties of dilation and erosion:

- Dual operators \((\varepsilon(f))^c = \delta(f^c)\)

Both dilation and erosion are increasing operators:

- \(f \leq g \rightarrow \begin{cases} \varepsilon(f) \leq \varepsilon(g) \\ \delta(f) \leq \delta(g) \end{cases} \)
EROSION AND DILATION

If the origin $O$ is part of the structuring element:

- Dilation is extensive and erosion is anti-extensive

- $O \in S \Rightarrow \{\varepsilon(f) \leq f, f \leq \delta(f)\}$

Separability: symmetrical structuring elements can be separated into one-dimensional parts:

- $S = \delta_{S_1}(S_2) \Rightarrow \begin{cases} \varepsilon_S(f) = \varepsilon_{S_1} (\varepsilon_{S_2}(f)) \\ \delta_S(f) = \delta_{S_1} (\delta_{S_2}(f)) \end{cases}$

- E.g. can be separated into dilation with and
QUESTIONS?
OPENING AND CLOSING
OPENING AND CLOSING

Opening: erosion followed by dilation:

- \( \gamma(f) = \delta(\varepsilon(f)) \)
- Small holes and cavities become bigger

Closing: dilation followed by erosion

- \( \varphi(f) = \varepsilon(\delta(f)) \)
- Removes small holes and cavities
OPENING AND CLOSING²

Original

Opening

Closing

3x3

7x7
OPENING AND CLOSING

Example:

Original | Opening | Closing
OPENING AND CLOSING

Properties of opening and closing

• Dual operators
• The opening is anti-extensive
• The closing is extensive
• Both opening and closing are idempotent (applying them twice gives the same result)
OPENING AND CLOSING

Exercise:

• Compute the opening and closing of the following image using a 3x3 square element

```
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
0 0 1 1 1 0 0 0
0 0 1 0 1 0 0 0
0 0 1 1 1 0 0 0
0 0 1 1 1 0 0 0
0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0
```
ASSIGNMENT
ASSIGNMENT

Ideas for the assignment can be checked until October 8

In general:

- Think of something that you could analyze in terms of shape, NOT color
- Gather a number of example pictures
- Develop your algorithms/approach
- Find out how well it works

Examples:

- Sea stars
- Traffic signs (of a particular shape)
- Certain candy
- Smart phones
GEODESIC OPERATIONS AND RECONSTRUCTION
GEODESIC OPERATIONS

The dilation is extensive:

- It “grows” at the boundaries
- Often we want to constrain the growth
- We could set some boundaries using control image $g$
- Similarly, for erosion $g$ is a “minimum image”

**Geodesic dilation** $\delta_g(f) = \min(\delta(f), g)$

**Geodesic erosion** $\varepsilon_g(f) = \max(\varepsilon(f), g)$

$g$ acts as a “mask”
GEODESIC OPERATIONS$^2$

Example:

\[ \delta_g(f) = \min(\delta(f), g) \]
GEODESIC OPERATIONS

Example:

step 1

step n
GEODESIC OPERATIONS

Iteration of geodesic dilation will always converge to stable result

- Maximum shape is $g$
- We call this a reconstruction
GEODESIC OPERATIONS

Opening by reconstruction of erosion:

• An erosion followed by a reconstruction (by dilation) with initial image
• To remove small structures without change in larger structures
• Size of structuring element determines how much will be removed
GEODESIC OPERATIONS

Original

Erosion

Geodesic dilation

Reconstruction
GEODESIC OPERATIONS

If we are interested in the small structures, not the large ones, we can just subtract the opening by reconstruction from the original.
GEODESIC OPERATIONS

We can also perform opening by reconstruction on grayscale images.

Original  Erosion (?)  Reconstruction
GEODESIC OPERATIONS

Reconstruction from markers: reconstruction from a few specifically selected points

Markers obtained from:

- User clicks
- Image processing
GEODESIC OPERATIONS

Also in grayscale images, we can extract or remove objects using markers.
Another application: removing objects that intersect with the border

Use boundary as marker image

- Subtract reconstruction from original
QUESTIONS?
EXERCISES
Next lecture is about:

- Mathematical morphology – part 2 (chapter 6 of reader)
- Friday September 25, 09:00-10:45 (Ruppert-040)

Suggested “homework”:

- Find a partner for the assignment
- Check out the C# software framework for the assignment
- You can check with me whether your proposal is suitable (until October 8!)