Overview

• Basic Modal Logic
• Relevant Forms of Modal Logic
  - Dynamic logic
  - Temporal logic
  - Epistemic logic
  - Deontic logic
  - Context logic
  - ...

Overview (ctd)

• Applications
  - Agent logics such as BDI-CTL
  - Common Knowledge & Joint/Collective Intentions
  - ...

Basic Modal Logic

• Expresses ‘intensional’ (context/situation-sensitive) notions such as
  - Knowledge
  - Belief
  - Obligation
  - Action
  - Time

Modal language

• Propositional logic
• Extended with modal operators
  - \( \Box \phi \): it is necessary that \( \phi \)
  - \( \Diamond \phi \): it is possible that \( \phi \)

Semantics: Kripke models

• Kripke model \( M = (S, \pi, R) \)
  - \( S \) is a set of worlds (states)
  - \( \pi \) is a truth assignment function
  - \( \pi: S \times AT \rightarrow \{tt, ff\} \)
  - \( R \subseteq S \times S \) is an accessibility relation
Kripke models

Interpretation
• Like classical propositional logic
• But now relative to a model and a world (state):
  \[ M, s \models \phi \]
• E.g.
  \[ M, s \models p \iff s(s, p) = \text{tt} \]
  \[ M, s \models \phi \land \psi \iff M, s \models \phi \text{ and } M, s \models \psi \]

Interpretation of \(\Box\) and \(\Diamond\)
• \( M, s \models \Box \phi \iff M, t \models \phi \) for every \( t \) such that \( R(s, t) \)
• \( M, s \models \Diamond \phi \iff M, t \models \phi \) for some \( t \) such that \( R(s, t) \)

Interpretation of \(\Box\)

Interpretation of \(\Diamond\)

Validity in modal logic
• \( \phi \) is valid in a model \( M = (S, R, \mu) \)
  (denoted \( M \models \phi \)) \iff
  \( M, s \models \phi \) for all \( s \in S \).
• \( \phi \) is valid (denoted \( \models \phi \)) \iff
  \( M \models \phi \) for all Kripke models \( M \).
• Sometimes we need validity wrt subclasses of models:
  \( \phi \) is valid wrt class \( \mathcal{C} \) (denoted \( \models \phi \) \iff
  \( M \models \phi \) for all Kripke models \( M \in \mathcal{C} \).
Basic Modal Logic: system \( K(m) \)

- As usual in logic we can try to axiomatize validities.

- Axioms:
  - All (or enough) propositional tautologies
  - \( \Box \phi \land \Box \psi \Rightarrow \Box (\phi \land \psi) \) (K axiom)

- Rules:
  - \( \Box \phi \Rightarrow \phi / \Box \phi \) (modus ponens)
  - \( \phi \lor \Box \phi \Rightarrow \phi \) (necessitation rule)

Caution!!

- NB. Distinguish the necessitation rule
  - \( \phi \Rightarrow \Box \phi \)
  - from the invalid assertion
  - \( \phi \Rightarrow \Box \phi \)

Derivability in \( K \)

- A derivation of a formula \( \psi \) is a finite sequence of formulas \( \phi_1, \phi_2, ..., \phi_n = \psi \), where each \( \phi_i \), for \( 1 \leq i \leq n \), is either an instance of the axioms (or rather axiom schemes), or the conclusion of one of the rules of which the premises have been derived already, i.e. appear as \( \phi_j \) in the sequence with \( j < i \).

- When we can derive an epistemic formula \( \psi \) by using the axioms and rules of \( K(m) \), we write \( K(m) \vdash \psi \).

System \( K(m) \)

- System \( K(m) \) is sound and complete, i.e.
  - \( \vdash \psi \Leftrightarrow K(m) \vdash \psi \)

  
  This means that exactly all valid modal assertions can be obtained by derivations in system \( K(m) \).

Some (non)theorems in \( K \)

- \( \vdash \Box \phi \land \psi \Rightarrow (\Box \phi \land \Box \psi) \)
- \( \vdash \Box (\phi \lor \psi) \Rightarrow (\Box \phi \lor \Box \psi) \)
- \( \vdash \Box (\phi \land \psi) \Rightarrow (\Box \phi \land \Box \psi) \)
- \( \vdash \phi (\phi \lor \psi) \Rightarrow \phi \lor \Box \psi \)
- \( \vdash \Box (\phi \lor \psi) \Rightarrow (\Box \phi \lor \Box \psi) \)
- \( \vdash \Box (\phi \land \psi) \Rightarrow (\Box \phi \land \Box \psi) \)

Application: Dynamic Logic

- An example of an (indexed) version of system \( K \) is dynamic logic, where the \( \Box \) modality is associated with the execution results of a program / action
  - \( \Box_\alpha \), normally written \([\alpha]\)
Dynamic Logic

- **Syntax**
  - Operator $[u]$ with reading:
  - $[u]q$: after execution of $u$ it holds (nec.) that $q$
  - $\langle [u] \rangle q$: after execution of $u$ it holds (poss.) that $q$

- **Semantics**
  - Accessibility relation $R_u$ for every action $u$.
    - $R_u = R_u \circ R_u$
    - $R_u = R_u \cup R_v$
    - $R_v = R_v \circ R_u$

Dynamic Logic

- **Interpretation formulas**
  - $M, s \vDash [u] \psi$: for all $s'$ with $R_u(s, s')$: $M, s' \vDash \psi$
  - $M, s \vDash <u> \psi$: for some $s'$ with $R_u(s, s')$: $M, s' \vDash \psi$

Special properties of accessibility relations

- $R$ is reflexive if $\forall s \in S \ (s, s) \in R$.
- $R$ is transitive if $\forall s, t, u \in S \ ((s, t) \in R \land (t, u) \in R) \Rightarrow (s, u) \in R$.
- $R$ is symmetrical if $\forall s, t \in S \ ((s, t) \in R) \Rightarrow (t, s) \in R$.
- $R$ is euclidean if $\forall s, t, u \in S \ ((s, t) \in R \land (t, u) \in R) \Rightarrow (s, u) \in R$.
- $R$ is serial if $\forall s \in S \exists t \in S \ (s, t) \in R$.
- $R$ is an equivalence relation if $R$ is reflexive, transitive and symmetrical

Special classes of models

- $\mathcal{K}_m$ is the class of all reflexive Kripke models with $m$ agents.
- $\mathcal{S}_m$ is the class of all reflexive-transitive Kripke models with $m$ agents.
- $\mathcal{S}_m$ is the class of all Kripke models with $m$ agents with accessibility relations that are equivalence relations.
- $\mathcal{K}_m$ is the class of all Kripke models with $m$ agents with serial accessibility relations.
- $\mathcal{K}_m$ is the class of all Kripke models with $m$ agents with serial, transitive and euclidean accessibility relations.

Systems $T$, $S_4$, $S_5$, $KD$, $KD_{45}$

- $T_m = K_m + \text{axiom } \Box \Box \phi \rightarrow \phi$
- $S_4(m) = T_m + \text{axiom } \Box \Box \phi \rightarrow \Box \Box \phi$
- $S_5(m) = S_4(m) + \text{axiom } \Diamond \Box \phi \rightarrow \Diamond \Box \phi$
- $KD_m = K_m + \text{axiom } \Box \Box \phi \rightarrow \Box \Box \phi$
- $KD_{45}(m) = K_m + \text{axioms } \Box \Box \phi \rightarrow \Box \Box \phi$, $\Box \phi \rightarrow \Box \Box \phi$
Alternative formulation of the 5-axiom

• \( \neg \Box \phi \rightarrow \Box \neg \Box \phi \)

Rewrite:

• \( \Box \neg \phi \rightarrow \Box \Box \neg \phi \)

Substitute \( \psi \) for \( \neg \phi \):

• \( \Box \psi \rightarrow \Box \Box \psi \)

Substitute \( \phi \) for \( \psi \):

• \( \Box \phi \rightarrow \Box \Box \phi \)

Soundness & completeness of T, S4, S5, KD, KD45

\[
\begin{align*}
T_{(m)} \vdash \phi & \iff T_{(m)} \models \phi \\
S4_{(m)} \vdash \phi & \iff S4_{(m)} \models \phi \\
S5_{(m)} \vdash \phi & \iff S5_{(m)} \models \phi \\
KD_{(m)} \vdash \phi & \iff KD_{(m)} \models \phi \\
KD45_{(m)} \vdash \phi & \iff KD45_{(m)} \models \phi
\end{align*}
\]

Deontic logic

• The system KD is also known as SDL (standard deontic logic)
• Deontic logic is the logic of obligation, prohibition and permission, or rather: the logic of ideal vs actual situations

Deontic logic

• Prop. Calculus (including MP)
• \( (O\phi \land O(\psi \rightarrow \phi)) \rightarrow O\psi \) (K axiom)
• \( \phi / O\phi \) (necessitation rule)
• \( \neg O\bot \) (D-axiom: obligation is consistent)
• \( F\psi \leftrightarrow O\neg\psi \) (forbidden is obliged to not)
• \( P\psi \leftrightarrow \neg F\psi \) (permitted is not forbidden)

Temporal Logic

• Basic linear-time logic (LTL)
• Time as accessibility relation
  - Reflexive
  - Transitive

Basic LTL

• Viewed in this way:
  - LTL = S4(\(\Box\))
  - \( \Box \) stands for "always in the future"
  - By convention the present included in the future
Basic LTL
- Prop. Calculus (including MP)
- \((\Box \phi \land \Box(\phi \rightarrow \psi)) \rightarrow \Box \psi\) (K axiom)
- \(\phi / \Box \phi\) (necessitation rule)
- \(\Box \phi \rightarrow \psi\) (always implies now)
- \(\Box \phi \rightarrow \Box \Box \phi\) (always implies always always)

Epistemic & Doxastic Logic
- For knowledge we take the relation \(R_i:\)
  - reflexive, (transitive) and euclidean
  - (i.e. an equivalence relation)
- For belief we take the relation \(R_i:\)
  - serial, transitive and euclidean

Basic Epistemic Logic: \(S5(\text{m})\)
- Prop. Calculus (including MP)
- \((K_i \phi \land K_i(\phi \rightarrow \psi)) \rightarrow K_i \psi\) (K axiom)
- \(\phi / K_i \phi\) (necessitation rule)
- \(K_i \phi \rightarrow \psi\) (knowledge is true)
- \(K_i \phi \rightarrow K_i K_i \phi\) (positive introspection)
- \(\neg K_i \phi \rightarrow K_i \neg K_i \phi\) (negative introspection)

Interpretation of \(K_i\)

Basic Doxastic Logic: \(KD45(\text{m})\)
- Prop. Calculus (including MP)
- \((B_i \phi \land B_i(\phi \rightarrow \psi)) \rightarrow B_i \psi\) (K axiom)
- \(\phi / B_i \phi\) (necessitation rule)
- \(\neg B_i \bot\) (belief is consistent)
- \(B_i \phi \rightarrow B_i B_i \phi\) (positive introspection)
- \(\neg B_i \phi \rightarrow B_i \neg B_i \phi\) (negative introspection)

More advanced applications
- Context logic
- Common knowledge & belief
- Collective intentions
- (Extended) LTL
- CTL (tree logic, branching-time)
- BDI-CTL