Today: Vectors and vector algebra
Ray tracing – part I of the course
Ray tracing – part I of the course

• Why vectors?
  – you need to shoot a lot of such rays!
  – vectors are the vehicles you need for ray tracing
And vectors as vehicles for....

- Define a virtual scene
- Define a camera direction
- Trace a bullet around a scene
- Line-of-sight queries
- And greetings from the gaming community (NVIDIA, Microsoft)....
  - https://blogs.nvidia.com/blog/2018/03/21/
    (epic-games-reflections-ray-tracing-offers-peek-gdc)
  - https://www.youtube.com/watch?v=-zW3Ghz-WQw
  - https://www.youtube.com/watch?v=81E9yVU-KB8
 Scalars (before we talk about vectors!)

- Quantities that can be described by a magnitude (i.e., a single number)
Scalars

- Quantities that can be described by a magnitude (i.e., a single number)
  - this sack of potatoes weighs 5 kilos
  - distance between Utrecht and Amsterdam is 40.5 kms
  - the car is travelling with speed 50 km/h
  - numbers like $\pi = 3.14159\ldots$, $e = 2.71818\ldots$, $1/3$, $-1/\sqrt{2}$ etc.

- On a computer: int, float, double
Vectors

- Quantities that have not only a magnitude but also a direction
Vectors

- Quantities that have not only a magnitude but also a direction
  - Utrecht-Amsterdam example (40.5 kms in-between)
  - the velocity of an airplane
Vectors

- Quantities that have not only a magnitude but also a direction
  - Utrecht-Amsterdam example (40.5 kms in-between)

- One way to represent the U-A vector:
  - start at U and end at A; vector (the arrow!) spans the two
  - start-point (U): move 40.5 kms in 24° west of north
Vectors

- Quantities that have not only a magnitude but also a direction
  - Utrecht-Amsterdam example (40.5 kms in-between)

- Equivalent second way to represent the U-A vector:
  - start-point (U): move 37 kms north and 16.47 kms west
    ("north" and "west" are reference directions)
Reference directions ⇒ a co-ordinate system

- Number of reference directions = dimensionality of space
  - $d$-dimensional space $\equiv \mathbb{R}^d$; 2D $\equiv \mathbb{R}^2$, 3D $\equiv \mathbb{R}^3$ ...
Reference directions $\Rightarrow$ a co-ordinate system

- Number of reference directions = dimensionality of space
  $d$-dimensional space $\equiv \mathbb{R}^d$; 2D $\equiv \mathbb{R}^2$, 3D $\equiv \mathbb{R}^3$ ...

- Cartesian co-ordinate system
  in 3D:
  (reference directions $\perp$ to each other)
Reference directions ⇒ a co-ordinate system

- **Cartesian co-ordinate system**
  - in 3D:
    - A point $P$ is represented by $(x, y)$ co-ordinates in two dimensions
    - by $(x, y, z)$ co-ordinates in three-dimensions
    - by $(x_1, x_2, \ldots, x_d)$ co-ordinates in $d$ dimensions

- **Origin of a co-ordinate system**: all entries of $P$ are zero
Reference directions $\Rightarrow$ a co-ordinate system

- Co-ordinate system does not have to be orthogonal/Cartesian!
Reference directions ⇒ a co-ordinate system

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Reference directions ⇒ a co-ordinate system

- Co-ordinate system does not have to be orthogonal/Cartesian!

- There are advantages for Cartesian co-ordinate systems (later)
Q. Latitude-longitude: is it a co-ordinate system?
A. Latitude-longitude: It is a co-ordinate system
Latitude-longitude: It is a co-ordinate system

Q. Is it orthogonal?
Latitude-longitude: It is a co-ordinate system

A. It is (locally) orthogonal
A point in a co-ordinate system

- Is represented as an array on a computer
A point in a co-ordinate system

- Is represented as an array on a computer
  - example in 5 dimensions \( (d = 5) \): \( P = (73, 98, 86, 61, 96) \)
A vector in a co-ordinate system

- Like Utrecht → Amsterdam (37 kms N, 16.47 kms W)
A vector in a co-ordinate system

- Like Utrecht → Amsterdam (37 kms N, 16.47 kms W)
  - an example vector in 5 dimensions \((d = 5)\): \(\vec{v} = (73, 98, 86, 61, 96)\)
    is also represented as an array on the computer!

<table>
<thead>
<tr>
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<th>73</th>
</tr>
</thead>
<tbody>
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<td>arr[1]</td>
<td>98</td>
</tr>
<tr>
<td>arr[2]</td>
<td>86</td>
</tr>
<tr>
<td>arr[3]</td>
<td>61</td>
</tr>
<tr>
<td>arr[4]</td>
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A vector in a co-ordinate system

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- So... what is the difference between a point and a vector?
A point (on a co-ordinate system) vs a vector (e.g., in 3D)
A point (on a co-ordinate system) vs a vector (e.g., in 3D)

- A vector does not specify the starting point!
  - it only specifies the length and the direction of the arrow
Summary so far...

- Scalar: Quantity represented by a magnitude (a single number)

- Vector: Quantity requiring a magnitude and a direction
  - to represent it, we need a co-ordinate system
    (a) does not have to be Cartesian
    (b) we will use Cartesian unless otherwise stated
  - number of reference directions = number of spatial dimensions
  - A point $P$ is represented by $(x_1, x_2, \ldots, x_d)$ in $d$ spatial dimensions
    [by $(x, y)$ in 2D and by $(x, y, z)$ in 3D]
  - both points and vectors are represented by an array on a computer
    (a vector is however fundamentally different entity than a point)
Point and vector representation

• Point $P$: $(x_1, x_2, \ldots, x_d)$ in $d$ spatial dimensions
  
  - by $(x, y)$ in 2D and by $(x, y, z)$ in 3D

  \[
  \begin{bmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_d
  \end{bmatrix}
  \]

• Vector $\vec{v}$: in $d$ spatial dimensions: vector notation

  \[
  \begin{bmatrix}
  v_x \\
  v_y \\
  \vdots \\
  v_z
  \end{bmatrix}
  \]

  - by $\begin{bmatrix} v_x \\ v_y \end{bmatrix}$ in 2D and by $\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ in 3D

  - the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ spans the origin and the point $(x, y, z)$ in 3D
Vector addition

(Only for vectors of the same dimension!)

- Vectors \( \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_d \end{bmatrix} \) and \( \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix} \)

- Vector \( \vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ . \\ . \\ u_d + v_d \end{bmatrix} \)
Vector subtraction

(Only for vectors of the same dimension!)

- Vectors \( \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_d \end{bmatrix} \) and \( \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix} \)

- Vector \( \vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ . \\ . \\ u_d - v_d \end{bmatrix} \)

- \( \vec{u} - \vec{v} = 0 \Rightarrow \vec{u} = \vec{v} \Rightarrow u_1 = v_1, u_2 = v_2, \ldots, u_d = v_d \)
Vector addition and subtraction: example

• Addition: \( \vec{a} = \vec{u} + \vec{v} + \vec{w} \)

Example: \( \vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \)

• Subtraction: \( \vec{a} - \vec{w} \); reverse the direction of \( \vec{w} \) and vector-add to \( \vec{a} \) (i.e., to get \(-\vec{w}\) simply reverse the arrow)
Scalar multiplication of a vector

- Vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix}$, scalar $\lambda$

$\lambda \vec{v} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_d \end{bmatrix}$
Magnitude (length, or norm) of a vector

(Formulas below holds for Cartesian co-ordinate system only!)

Vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$; magnitude (norm) $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_d^2}$

2D: $||\vec{v}|| = \sqrt{v_x^2 + v_y^2}$; 3D: $||\vec{v}|| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

($||\vec{v}||$ is the length of the arrow)
Magnitude (length/norm) of a vector, and unit vector

(Formulas below holds for Cartesian co-ordinate system only!)

- Vector \( \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix} \); magnitude (norm) \( ||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_d^2} \)

\( \text{2D: } ||\vec{v}|| = \sqrt{v_x^2 + v_y^2}, \text{ 3D: } ||\vec{v}|| = \sqrt{v_x^2 + v_y^2 + v_z^2} \)

(\( ||\vec{v}|| \) is the length of the arrow)

- Corresponding unit vector \( \hat{v} = \frac{1}{||\vec{v}||} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix} \); you can confirm \( ||\hat{v}|| = 1 \)

(this process is called normalisation)
Magnitude (length) of a vector, unit and basis vectors

(Formulas below hold for Cartesian co-ordinate system only!)

- Vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$; magnitude (norm) $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_d^2}$

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  (this process is called normalisation)

  - unit vectors in reference directions $\hat{x}_1, \hat{x}_2, \ldots$ are the basis vectors (e.g., $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the basis vectors in 3D; in vector notation?)
Magnitude (length) of a vector, unit and basis vectors

(Formulas below holds for Cartesian co-ordinate system only!)

- Vector \( \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix} \); magnitude (norm) \( ||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_d^2} \)

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  - unit vectors in reference directions \( \hat{x}_1, \hat{x}_2, \ldots \) are the basis vectors

- Why do we need a Cartesian co-ordinate system?
Pythagoras’ theorem and elementary trigometry
(works for Cartesian co-ordinate system only!)

- In 2D: basis vectors $\hat{x}, \hat{y}$; $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$; $||\vec{v}||^2 = v_x^2 + v_y^2$ (Pythagoras)
Pythagoras’ theorem

\[ a^2 + b^2 = c^2 \]

or

\[ c = \sqrt{a^2 + b^2} \]
Pythagoras’ theorem

(below it works for Cartesian co-ordinate system only!)

- In 2D: basis vectors $\hat{x}, \hat{y}$; $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$; $||\vec{v}||^2 = v_x^2 + v_y^2$

  e.g., $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $||\vec{v}|| = \sqrt{3^2 + 4^2} = 5$
Pythagoras’ theorem

(below it works for Cartesian co-ordinate system only!)

- In 2D: basis vectors \( \hat{x}, \hat{y} \); \( \vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \); \( ||\vec{v}||^2 = v_x^2 + v_y^2 \)

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- Pythagoras in \( d \) dimensions: \( ||\vec{v}||^2 = v_1^2 + v_2^2 + \ldots + v_d^2 \)

- Null vector: vector of magnitude zero; \( v_1 = v_2 = \ldots = v_d = 0 \)
Cartesian co-ordinate system vectors and trigonometry

- In 2D: basis vectors $\hat{x}, \hat{y}$; $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$; $||\vec{v}||^2 = v_x^2 + v_y^2$ (Pythagoras)

- $\cos \theta = \frac{v_x}{||\vec{v}||}$, $\sin \theta = \frac{v_y}{||\vec{v}||}$, $\tan \theta = \frac{v_y}{v_x}$

unit circle
Cartesian co-ordinate system vectors and trigonometry

- In 2D: basis vectors $\hat{x}, \hat{y}$; $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$; $||\vec{v}||^2 = v_x^2 + v_y^2$ (Pythagoras)

- $\cos \theta = \frac{v_x}{||\vec{v}||}$, $\sin \theta = \frac{v_y}{||\vec{v}||}$, $\tan \theta = \frac{v_y}{v_x}$

- $v_x^2 + v_y^2 = ||\vec{v}||^2 \Rightarrow \sin^2 \theta + \cos^2 \theta = 1$

Q. What is $\hat{v}$ in terms of $\theta$? (think in terms of the unit circle!)

<table>
<thead>
<tr>
<th>Angle (°)</th>
<th>Degrees</th>
<th>(θ)</th>
<th>sin(θ)</th>
<th>cos(θ)</th>
<th>tan(θ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\sqrt{3}$</td>
<td></td>
</tr>
<tr>
<td>90°</td>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>0</td>
<td>Not Defined</td>
<td></td>
</tr>
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Cartesian co-ordinate system vectors and trigonometry

- In 2D: basis vectors \( \hat{x}, \hat{y} \); \( \vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} \); \( ||\vec{v}||^2 = v_x^2 + v_y^2 \) (Pythagoras)

- \( \cos \theta = \frac{v_x}{||\vec{v}||}, \sin \theta = \frac{v_y}{||\vec{v}||}, \tan \theta = \frac{v_y}{v_x} \)

- \( v_x^2 + v_y^2 = ||\vec{v}||^2 \) \Rightarrow \( \sin^2 \theta + \cos^2 \theta = 1 \)

A. \( \hat{v} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \)

<table>
<thead>
<tr>
<th>Angle ( \theta )</th>
<th>( \sin(\theta) )</th>
<th>( \cos(\theta) )</th>
<th>( \tan(\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0^\circ )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( 30^\circ )</td>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
</tr>
<tr>
<td>( 45^\circ )</td>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
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The second summary...

- Vector operations
  - addition and subtraction (requires same dimensionality)
  - scalar multiplication
  - magnitude/length/norm of a vector; unit, null and basis vectors
  - Pythagoras theorem
  - elementary trigonometry: definitions of sin, cos, tan
The second summary...

● Vector operations
  – addition and subtraction (requires same dimensionality)
  – scalar multiplication
  – magnitude/length/norm of a vector; unit, null and basis vectors
  – Pythagoras theorem
  – elementary trigonometry: definitions of \( \sin \), \( \cos \), \( \tan \)

● Next class: vector algebra (contd.), and shooting rays to objects in 2D
Finally, references

- Book chapter 2: Miscellaneous Math
  - Sec. 2.3
  - Secs. 2.4.1-2.4.2, 2.4.5