\[
\begin{align*}
\vec{a} & = \lambda_1 \hat{x} + \lambda_2 \hat{y} + \lambda_3 \hat{z} \\
& = \lambda_1' \hat{x'} + \lambda_2' \hat{y'} + \lambda_3' \hat{z'}
\end{align*}
\]

How are \( \lambda_1' \) and \( \lambda_2' \) related to \( \lambda_1 \) and \( \lambda_2 \)?

To determine things like this we need to invest in matrices.

A vector in \( d \) dimensions is a \( d \)-tuple of numbers or scalar variables.

\[
\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ in } 3d
\]

An \( m \times n \) matrix is an array of \( mn \) scalar values sorted in \( m \) rows and \( n \) columns:

\[
A = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
\text{ (matrix dimension is } m \times n)\\

\( a_{ij} \) are the matrix elements or matrix coefficients.

\( i \) is the row number, \( j \) is the column number.

\( m = n \) is called a square matrix.

**Addition of matrices**

\[
A = \{a_{ij}\} \quad B = \{b_{ij}\} \quad A + B = \{a_{ij} + b_{ij}\}
\]

You can only add matrices of the same dimensions.

\[
\begin{pmatrix}
  2 & 3 \\
  5 & 4
\end{pmatrix}
+ \begin{pmatrix}
  -1 & 2 \\
  2 & -2
\end{pmatrix}
= \begin{pmatrix}
  2-1 & 3+2 \\
  5+2 & 4-2
\end{pmatrix}
= \begin{pmatrix}
  1 & 5 \\
  7 & 2
\end{pmatrix}
\]

**Multiplication by a scalar**

\[
A = \{a_{ij}\} \quad B = \lambda A = \{b_{ij}\} = \{\lambda a_{ij}\}
\]

\[
2 \begin{pmatrix}
  1 & 5 \\
  4 & 3
\end{pmatrix}
= \begin{pmatrix}
  2 & 10 \\
  8 & 6
\end{pmatrix}
\]
Multiplication by matrices

A is a matrix of dimension m x n
B is a matrix of dimension m x k
C = AB is a matrix of dimension m x k

\[ c_{ij} = \sum_{jk} a_{ij} b_{jk} \]

\[
\begin{pmatrix}
2 & 6 & 1 \\
5 & 2 & 4
\end{pmatrix}
\begin{pmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{pmatrix}
= 
\begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix}
\]

\[
c_{11} = \sum_{j=1}^{3} a_{1j} b_{j1} = 2 \times 1 + 6 \times 2 + 1 \times 3 = 17
\]
\[
c_{12} = \sum_{j=1}^{3} a_{1j} b_{j2} = 2 \times 4 + 6 \times 5 + 1 \times 6 = 44
\]
\[
c_{21} = \sum_{j=1}^{3} a_{2j} b_{j1} = 5 \times 1 + 2 \times 2 + 4 \times 3 = 21
\]
\[
c_{22} = \sum_{j=1}^{3} a_{2j} b_{j2} = 5 \times 4 + 2 \times 5 + 4 \times 6 = 54
\]

Special matrices

- **Diagonal matrix**
  \[
  A = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \sqrt{3} & 0 \\
  0 & 0 & -2/3
  \end{pmatrix}
  \]

- **Identity matrix**
  \[
  I = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
  \end{pmatrix}
  \]

- **Zero matrix**
  \[
  \phi = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{pmatrix}
  \]

Multiplication by special matrices

- \( \phi A = A \phi = \phi \)
- \( AI = IA = A \)

Distributive:

- \( A(B+C) = AB + AC \)
- \( (A+B)C = AC + BC \)

Associative:

- \( (AB)C = A(BC) \)

\( AB \neq BA \)!!
Some further operations/properties/definitions for matrices

### Transpose of a matrix $A$ is $A^T$

$$A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix}$$

$A^T = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{14} & a_{24} & a_{34} \\
a_{14} & a_{24} & a_{34}
\end{pmatrix}$

### Column elements

Square matrix

$$A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}$$

$A^T = \begin{pmatrix}
a_{11} & a_{21} & a_{31} \\
a_{12} & a_{22} & a_{32} \\
a_{13} & a_{23} & a_{33}
\end{pmatrix}$

### Determinant of a matrix

- Only defined for square matrices
- $A = -5$ 1x1 matrix
- $\det A = |A| = 5$

Notation:

$$A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}$$

$$\det A = |A| = \begin{vmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{vmatrix}$$

Or, $A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}$

$$\det A = |A| = \begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}$$

### How do we calculate it?

1. **Define co-factors**: A factor of an element $a_{ij}$ is obtained by
2. **determinant of the $(n-1) \times (n-1)$ matrix when $i$ and $j$ row and column are excluded
3. **Multiplied by $(-1)^{i+j}$

**Example**: $A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}$

- $a_{11} = a_{22}$
- $a_{12} = -a_{21}$
- $a_{21} = -a_{12}$
- $a_{22} = a_{11}$
\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \]

cof. \[ a_{ij} = \begin{vmatrix} a_{11} & a_{12} \\ a_{32} & a_{33} \end{vmatrix} \]

\[ a_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \]

**Laplace's Expansion for Determinant**

**Take any row:**

**Row 1:**

\[ |A| = a_{11} \text{ cof } a_{11} + a_{12} \text{ cof } a_{12} = a_{11} a_{22} - a_{12} a_{21} \]

**Row 2:**

\[ |A| = a_{21} \text{ cof } a_{21} + a_{22} \text{ cof } a_{22} = -a_{21} a_{12} + a_{22} a_{11} \]

**Column 2:**

\[ |A| = a_{12} \text{ cof } a_{12} + a_{22} \text{ cof } a_{22} = -a_{12} a_{21} + a_{22} a_{11} \]

**Column 3:**

\[ |A| = a_{13} \text{ cof } a_{13} + a_{23} \text{ cof } a_{23} + a_{33} \text{ cof } a_{33} \]

\[ = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} + a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \]

\[ = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \]

\[ = a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} + a_{23} a_{31} a_{12} - a_{23} a_{11} a_{32} \]
Rule of Sarrus

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|A| = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32}$$

$$- a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Does NOT work for 2x2 matrices!!

Adjoint of a matrix (Sometimes called adjugate)

$$\text{adj}(A) = (\text{cof}(A))^T = \hat{A}$$

Inverse of a matrix

$$A^{-1} = \frac{\hat{A}}{|A|}; \text{ has the property } A^{-1}A = AA^{-1} = I$$

If in some cases $|A| = 0$ then $A$ is called a \textbf{Singular matrix}. Singular matrices cannot be inverted.
What does a determinant actually mean (geometric interpretation)

\[ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]
\[ \mathbf{v}_1 \quad \mathbf{v}_2 \]
\[ \mathbb{R}^2 \]

Oriented area in positive if \( \mathbf{v}_1 \rightarrow \mathbf{v}_2 \) is counterclockwise

If \( \mathbf{v}_1 \rightarrow \mathbf{v}_2 \) is clockwise

\[ \det (\mathbf{v}_1, \mathbf{v}_2) = -\det (\mathbf{v}_2, \mathbf{v}_1) \]

\[ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \]
\[ \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \]

Oriented volume is positive if \((\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\) forms a right-handed system

If any two of the vectors are linearly dependent, then the area or volume = 0 \( \Rightarrow \det \mathbf{A} = 0 \)

Matrix operations on vectors

\[ \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{md} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_d \end{pmatrix} = \begin{pmatrix} a_{11} \mathbf{v}_1 + \cdots + a_{1d} \mathbf{v}_d \\ \vdots \\ a_{m1} \mathbf{v}_1 + \cdots + a_{md} \mathbf{v}_d \end{pmatrix} \]

\[ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} a_{11} \mathbf{x} + a_{12} \mathbf{y} + a_{13} \mathbf{z} \\ a_{21} \mathbf{x} + a_{22} \mathbf{y} + a_{23} \mathbf{z} \\ a_{31} \mathbf{x} + a_{32} \mathbf{y} + a_{33} \mathbf{z} \end{pmatrix} \]

\[ \begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \\ \mathbf{z}' \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} \Rightarrow \mathbf{v}' = \mathbf{A} \mathbf{v} \]

Transformation of the vector and therefore point

\[ (x', y', z') \]

Scale \( \mathbf{v}_i \) by \( x \) and so on.
**Projection**

\[
\begin{pmatrix}
1 & 0 \\
0 & \frac{1}{V_1}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
x \\
0
\end{pmatrix} = x \bar{V}_1
\]

projection on to x-axis

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = x \bar{V}_1
\]

**Uniform Scaling in 2D**

\[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 2\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

**Non-Uniform Scaling**

\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
ax \\
by
\end{pmatrix}
\]
Reflection about \( y = x \) line
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
y \\
x
\end{pmatrix}
\]
\(\text{vector along } y = x \text{ line } \hat{v} = \begin{pmatrix} 1 \\
1 \end{pmatrix}\)
\[
a_{11} + a_{12} = 1
\]
\[
a_{21} + a_{22} = 1
\]

Swapping \( x \) and \( y \)

Shearing
\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
x + y \\
y
\end{pmatrix}
\]

Rotation
\[
\begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]
\(\cos \phi\) and \(\sin \phi\) are the components of the rotation vector along a circle.

Anti-clockwise rotation of object
\[
\begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
x' \\
y'
\end{pmatrix}
\]
Clockwise rotation of coordinate axes!
\[
\begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
x' \\
y'
\end{pmatrix}
\]

These belong to the so-called linear transformation of vectors;
\(\text{i.e., } T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})\)
\(T(c\vec{v}) = cT(\vec{v})\)
\(\text{or generally } T(c_1\vec{u} + c_2\vec{v}) = c_1T(\vec{u}) + c_2T(\vec{v})\)

for all scalars \(c_1\) and \(c_2\).

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix}
= \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]
\[
x'^2 + y'^2 = x^2 + y^2
\]

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix}
= \begin{pmatrix}
\frac{a}{b} & \frac{b}{a} \\
\frac{-b}{a} & \frac{a}{b}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{a} & 0 \\
0 & \frac{1}{b}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]
\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix}
= \begin{pmatrix}
\frac{a}{b} & \frac{b}{a} \\
\frac{-b}{a} & \frac{a}{b}
\end{pmatrix}
\begin{pmatrix}
\frac{a\phi}{b} & -\frac{b\phi}{a} \\
\frac{b\phi}{a} & \frac{a\phi}{b}
\end{pmatrix}
\]

\[
\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{a\phi}{b}\right)^2 + \left(\frac{b\phi}{a}\right)^2
\]