Graphics 2011/2012, 4th quarter

Lecture 5

Linear and affine transformations

Vector transformation: basic idea

Multiplication of an $n \times n$ matrix with a vector
(i.e. a $n \times 1$ matrix):

In 2D: $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$

In 3D: $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{pmatrix}$

The result is a (transformed) vector.

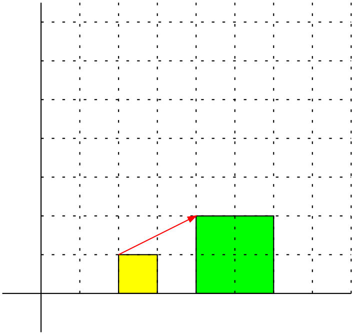
Example: scaling

To scale with a factor two with respect to the origin, we apply the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

to a vector:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 0y \\ 0x + 2y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$



Linear transformations

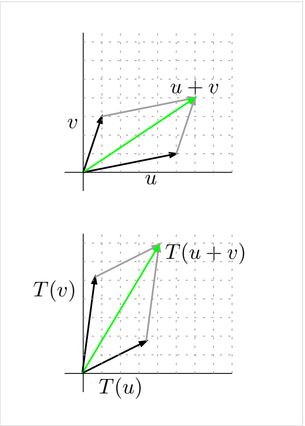
A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it satisfies

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$.
2. $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$ and all scalars c.

Or (alternatively), if it satisfies

$$T(c_1\vec{u} + c_2\vec{v}) = c_1T(\vec{u}) + c_2T(\vec{v})$$

for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and all scalars c_1, c_2.



⇝ Linear transformations can be represented by *matrices*

Example: scaling

We already saw scaling by a factor of 2.

In general, a matrix

$$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

scales the i-th coordinate of a vector by the factor $a_{ii} \neq 0$, i.e.

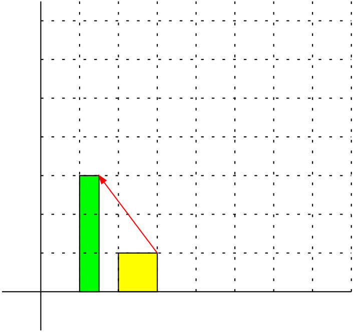
$$\begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + 0y \\ 0x + a_{22}y \end{pmatrix} = \begin{pmatrix} a_{11}x \\ a_{22}y \end{pmatrix}$$

Example: scaling

Scaling does *not* have to be uniform. Here, we scale with a factor one half in x-direction, and three in y-direction:

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x \\ 3y \end{pmatrix}$$

Q: what is the inverse of this matrix?



Example: scaling

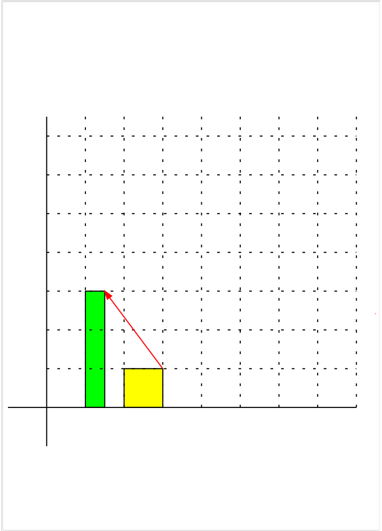
Using Gaussian elimination to calculate the inverse of a matrix:

$$\left(\begin{array}{cc|cc} \frac{1}{2} & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow$$

(Gaussian elimination)

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & \frac{1}{3} \end{array}\right)$$

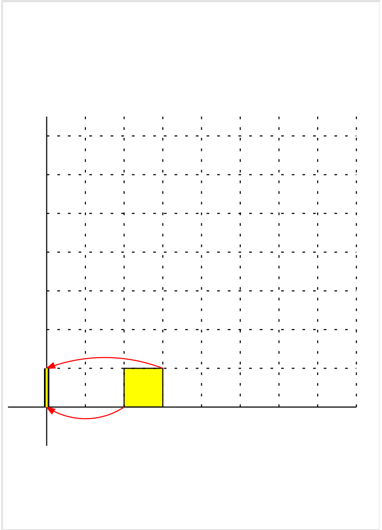


Example: projection

We can also use matrices to do orthographic projections, for instance, onto the Y-axis:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

Q: what is the inverse of this matrix?

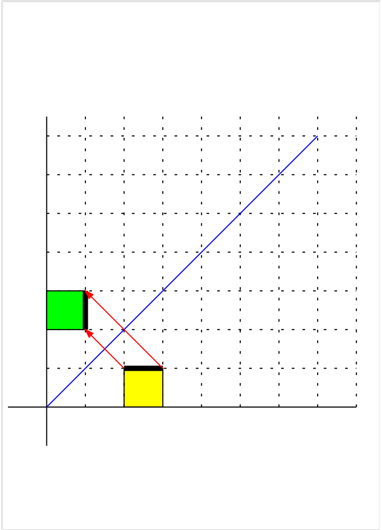


Example: reflection

Reflection in the line $y = x$ boils down to swapping x- and y-coordinates:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Q: what is the inverse of this matrix?



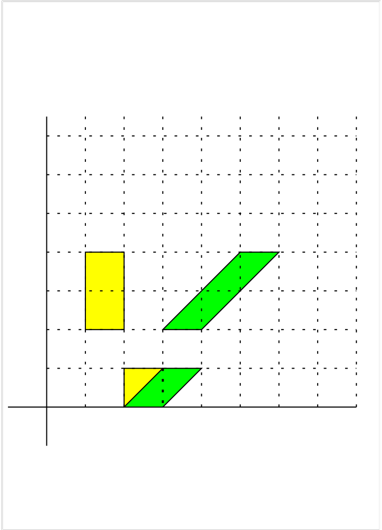
Example: shearing

Shearing in x-direction "pushes things sideways":

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix}$$

We can also "push things upwards" with

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x + y \end{pmatrix}$$



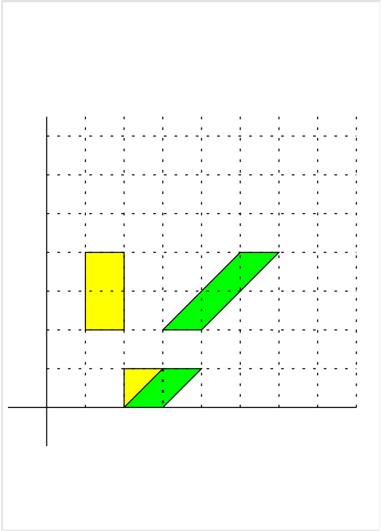
Example: shearing

General case for shearing in x-direction:

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + sy \\ y \end{pmatrix}$$

And its inverse operation:

$$\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - sy \\ y \end{pmatrix}$$



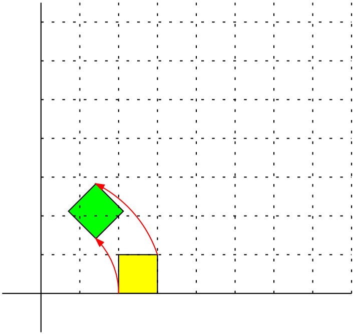
Example: rotation

To rotate 45° about the origin, we apply the matrix

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Note: $\frac{\sqrt{2}}{2} = \cos 45^\circ = \sin 45^\circ$, so this is the same as

$$\begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}$$



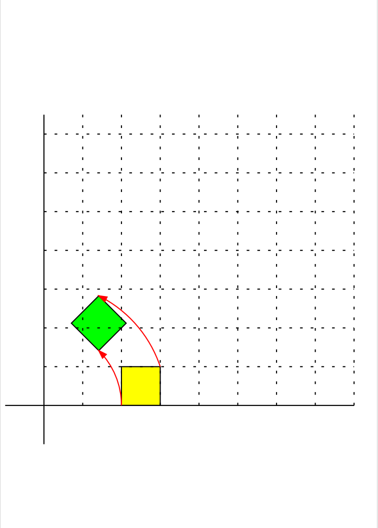
Example: rotation

General case for a counterclockwise rotation about an angle ϕ around the origin:

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

And for clockwise rotation:

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

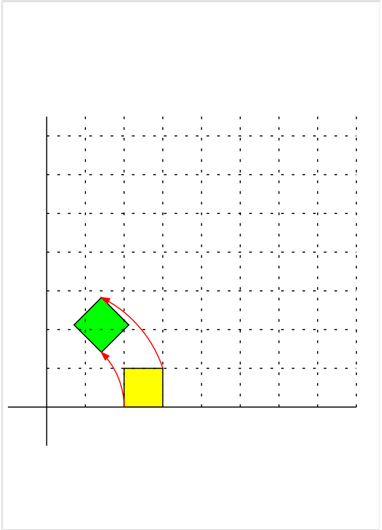


Finding matrices

Applying matrices is pretty straightforward, but how do we find the matrix for a given linear transformation?

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Q: what values do we have to assign to the a_{ij}'s to achieve a certain transformation?



Finding matrices

Let's apply some transformations to the base vectors $\vec{b_1} = (1, 0)$ and $\vec{b_2} = (0, 1)$ of the carthesian coordinates system, e.g.

Scaling (factors $a, b \neq 0$): $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}$

Shearing (x-dir., factor s): $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + sy \\ y \end{pmatrix}$

Reflection (in line $y = x$): $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

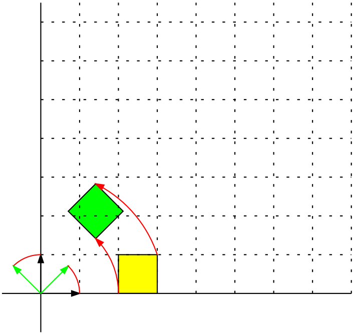
Rotation (countercl., $45°$): $\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} x - y \\ x + y \end{pmatrix}$

Finding matrices

Aha! The column vectors of a transformation matrix are the images of the base vectors!

That gives us an easy method of finding the matrix for a given linear transformation.

Why? Because matrix multiplication is a *linear* transformation.



Finding matrices

Remember: T is a linear transformation if and only if

$$T(c_1\vec{u} + c_2\vec{v}) = c_1T(\vec{u}) + c_2T(\vec{v})$$

Let's look at carthesian coordinates, where each vector $\vec{w}$ can be represented as a linear combination of the base vectors $\vec{b_1}$, $\vec{b_2}$:

$$\vec{w} = \begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

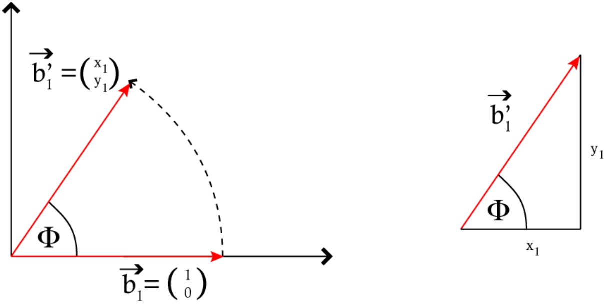
If we apply a linear transformation T to this vector, we get:

$$T(\begin{pmatrix} x \\ y \end{pmatrix}) = T(x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = xT(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + yT(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

Finding matrices: example

Rotation (counterclockwise, angle ϕ): $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

For example for the first base vector $\vec{b_1}$:



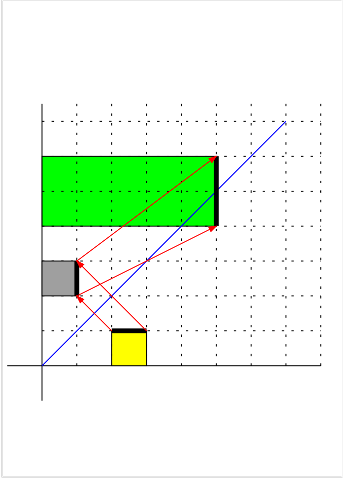
Example: reflection and scaling

Multiple transformations can be combined into one.

Here, we first do a reflection in the line $y = x$, and then we scale with a factor 5 in x-direction, and a factor 2 in y-direction:

$$\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Remember: matrix multiplication is *associative*, i.e. $A(Bx) = (AB)x$.



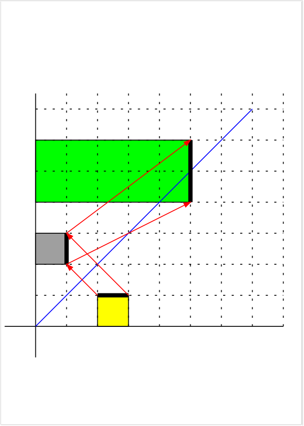
Example: reflection and scaling

But: Matrix multiplication is *not commutative*, i.e. in general $AB \neq BA$, for example:

$$\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5y \\ 2x \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ 5x \end{pmatrix}$$

Mind the order!

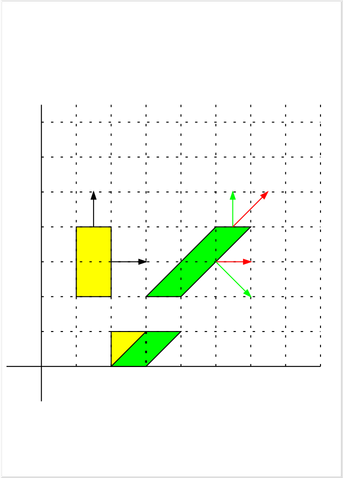


Transposing normal vectors

Unfortunately, normal vectors are not always transformed properly.

E.g. look at shearing, where tangent vectors are correctly transformed but normal vectors not.

To transform a normal vector $\vec{n}$ correctly under a given linear transformation A, we have to apply the matrix $(A^{-1})^T$. Why?



Transposing normal vectors

We know that tangent vectors are tranformed correctly: $A\vec{t} = \vec{t_A}$.

But this is not necessarily true for normal vectors: $A\vec{n} \neq \vec{n_A}$.

Goal: find matrix N_A that transforms $\vec{n}$ correctly, i.e. $N_A\vec{n} = \vec{n_N}$ where $\vec{n_N}$ is the correct normal vector of the transformed surface.

We know that

$$\vec{n}^T\vec{t} = 0.$$

Hence, we can also say that

$$\vec{n}^T I\vec{t} = 0$$

which is is the same as

$$\vec{n}^T A^{-1}A\vec{t} = 0$$

Transposing normal vectors

Because $A\vec{t} = \vec{t_A}$ is our correctly transformed tangent vector, we have

$$\vec{n}^T A^{-1} \vec{t_A} = 0$$

Because their scaler product is 0, $\vec{n}^T A^{-1}$ must be orthogonal to it. So, the vector we are looking for must be

$$\vec{n}_N^T = \vec{n}^T A^{-1}.$$

Because of how matrix multiplication is defined, this is a transposed vector. But we can rewrite this to

$$\vec{n}_N = (\vec{n}^T A^{-1})^T.$$

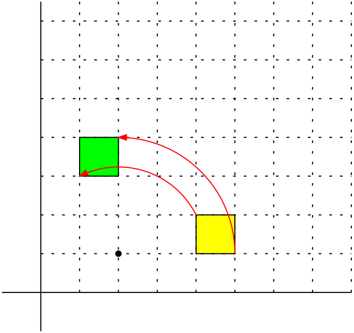
And if you remember that $(AB)^T = B^T A^T$, we get

$$\vec{n}_N = (A^{-1})^T \vec{n}$$

More complex transformations

So now we know how to determine matrices for a given transformation. Let's try another one:

Q: what is the matrix for a rotation of 90° about the point $(2, 1)$?

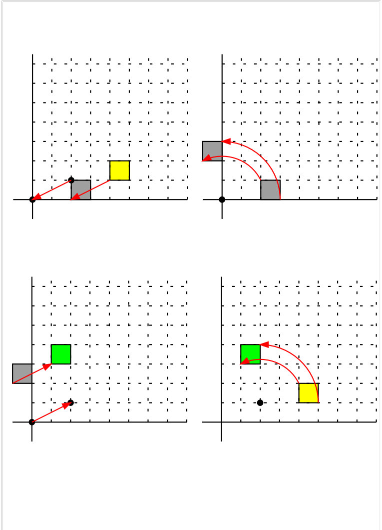


More complex transformations

We can form our transformation by composing three simpler transformations:

- Translate everything such that the center of rotation maps to the origin.
- Rotate about the origin.
- Revert the translation from the first step.

Q: but what is the matrix for a translation?



More complex transformations

Translation is not a linear transformation.

With linear transformations we get:

$$Ax = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x & + & a_{12}y \\ a_{21}x & + & a_{22}y \end{pmatrix}$$

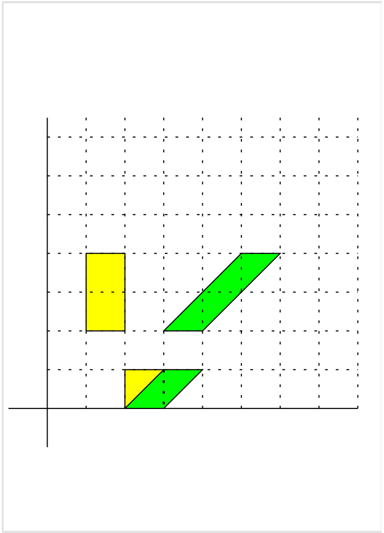
But we need something like:

$$\begin{pmatrix} x & + & x_t \\ y & + & y_t \end{pmatrix}$$

We can do this with a combination of linear transformations and translations called affine transformations.

Homogeneous coordinates

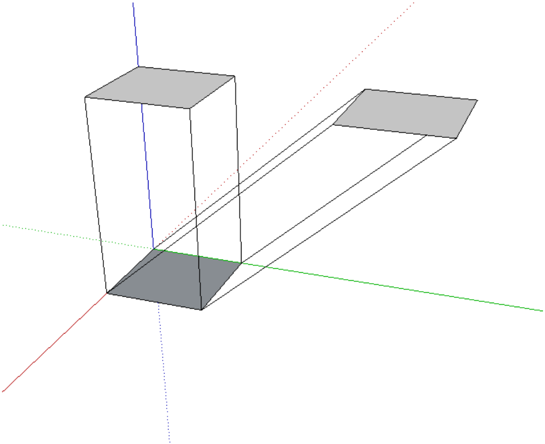
Observation:
Shearing in 2D smells a lot like translation in 1D . . .



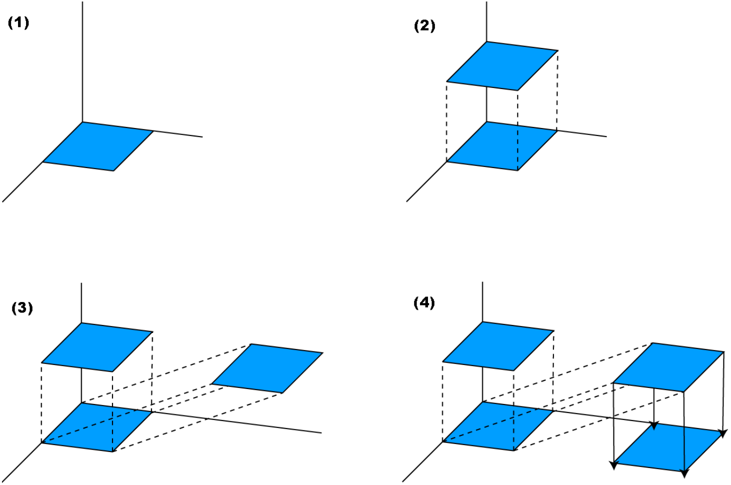
Homogeneous coordinates

. . . and shearing in 3D smells like translation in 2D

(. . . and shearing in 4D . . .)



Homogeneous coordinates in 2D: basic idea



Homogeneous coordinates in 2D: basic idea

We see: by adding a 3rd dimension to our 2D space, we can add a constant value to the first two coordinates by matrix multiplication.

$$M \begin{pmatrix} x \\ y \\ d \end{pmatrix} = \begin{pmatrix} x & + & x_t \\ y & + & y_t \\ d & & \end{pmatrix}$$

That's exactly what we want (for the first 2 coordinates).

But how does the matrix M look like?
And how are we dealing with this 3rd coordinate?

Homogeneous coordinates

Shearing in 3D based on the z-coordinate is a simple generalization of 2D shearing:

$$\begin{pmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ d \end{pmatrix} = \begin{pmatrix} x & + & x_t d \\ y & + & y_t d \\ & d & \end{pmatrix}$$

Notice that we didn't make any assumption about d, . . .

Homogeneous coordinates: matrices

. . . so we can use, for example, $d = 1$:

$$\begin{pmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x & + & x_t \\ y & + & y_t \\ & 1 & \end{pmatrix}$$

Homogeneous coordinates: points

Translations in 2D can be represented as shearing in 3D by looking at the plane $z = 1$.

By representing all our 2D points (x, y) by 3D vectors $(x, y, 1)$, we can translate them about (x_t, y_t) using the following 3D shearing matrix:

$$\begin{pmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + x_t \\ y + y_t \\ 1 \end{pmatrix}$$

Homogeneous coordinates: vectors

But can we translate a vector? No!
(Remember: vectors are defined by their length and direction.)

Hence, we have to represent a vector differently, i.e. by $(x, y, 0)$:

$$\begin{pmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Homogeneous coordinates

Affine transformations (i.e. linear transformations and translations) can be done with simple matrix multiplication if we add homogeneous coordinates, i.e. (in 2D):

- a third coordinate $z = 1$ to each location: $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$
- a third coordinate $z = 0$ to each vector: $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$
- a third row $(0 \ldots 0\ 1)$ to each matrix:

$$\begin{pmatrix} & & & * \\ & & & . \\ & \ldots & & . \\ & & & . \\ & & & * \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

Affine transformations: points vs. vectors

Scaling and moving a location/point (or "an object"):

$$\begin{pmatrix}1 & 0 & x_t\\ 0 & 1 & y_t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}s_1 & 0 & 0\\ 0 & s_2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ 1\end{pmatrix}=\begin{pmatrix}s_1 & 0 & x_t\\ 0 & s_2 & y_t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ 1\end{pmatrix}=\begin{pmatrix}s_1x+x_t\\ s_2y+y_t\\ 1\end{pmatrix}$$

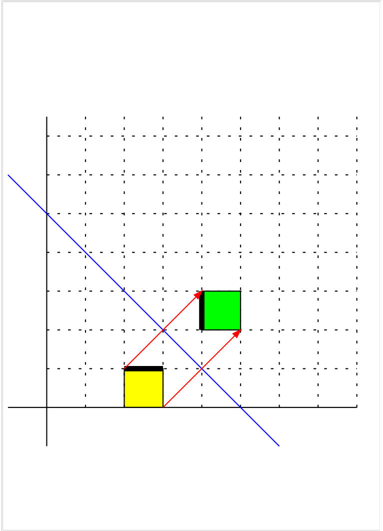
With the same matrix, we scale (but not move!) a vector:

$$\begin{pmatrix}s_1 & 0 & x_t\\ 0 & s_2 & y_t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ 0\end{pmatrix}=\begin{pmatrix}s_1x\\ s_2y\\ 0\end{pmatrix}$$

Affine transformations: example

What is the matrix for reflection in the line $y = -x + 5$?

Idea: move everything to the origin, reflect, and the move everything back.



Affine transformations: example

1. Translation of $y = -x + 5$ to $y = -x$

$$\begin{pmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x & + & x_t \\ y & + & y_t \\ & 1 & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & . \\ 0 & 1 & . \\ 0 & 0 & 1 \end{pmatrix}$$

2. Reflection on $y = -x$:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & . \\ -1 & 0 & . \\ . & . & . \end{pmatrix} \quad \begin{pmatrix} . & . & . \\ . & . & . \\ . & . & . \end{pmatrix}$$

3. Translate $y = -x$ back to $y = -x + 5$

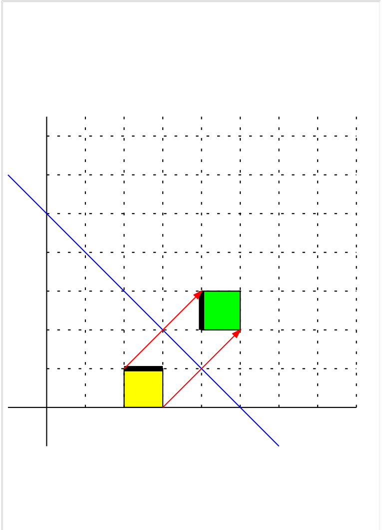
$$\begin{pmatrix} 1 & 0 & . \\ 0 & 1 & . \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} . & . & . \\ . & . & . \\ . & . & . \end{pmatrix}$$

Affine transformations

So, the matrix for reflection in the line $y = -x + 5$ is

$$\begin{pmatrix} 0 & -1 & 5 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

But what is the significance of the columns of the matrix?

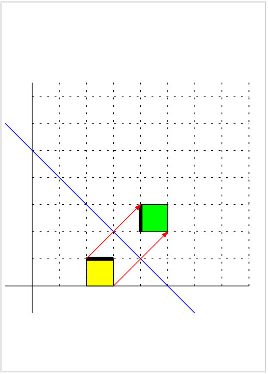


Affine transformations

The last column vector is the image of the origin after applying the *affine* transformation

(Remember: The other columns are the image of the base vectors under the *linear* transformation)

(Also remember: The coordinates in the last row are called *homogeneous coordinates*)



$$\begin{pmatrix} 0 & -1 & 5 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

Linear transformations in 3D

In general, linear transformations in 3D are a straightforward extension of their 2D counterpart.

We have already seen an example: shearing in $x-y-$direction

$$\begin{pmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{pmatrix}$$

Or in general

$$\begin{pmatrix} 1 & d_{x2} & d_{x3} \\ d_{y1} & 1 & d_{y3} \\ 0 & 0 & 1 \end{pmatrix}$$

Affine transformations in 3D

Homogeneous coordinates (and therewith affine transformations) in 3D are also a straightforward generalization from 2D.

We just have to use 4×4 matrices now:

$$\text{Matrices: } \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{points: } \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \text{ and vectors: } \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix}$$

Transformations in 3D

For scaling, we have three scaling factors on the diagonal of the matrix.

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix}$$

Shearing can be done in either x-, y-, or z-direction (or a combination thereof). For example, shearing in x-direction:

$$\begin{pmatrix} 1 & d_{x2} & d_{x3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + d_y y + d_z z \\ y \\ z \end{pmatrix}$$

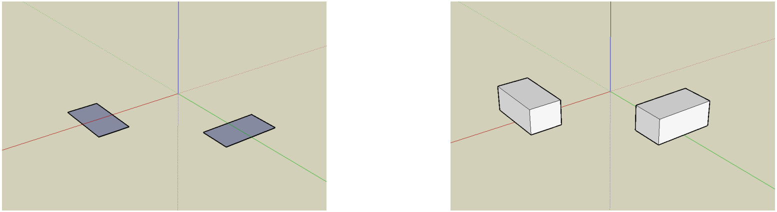
Transformations in 3D

We see: transformations in 3D are very similar to those in 2D.

- Scaling and shearing are straightforward generalizations of the 2D cases.
- Reflection is done with respect to *planes*
- Rotation is done about *directed lines*.

Transformations in 3D: rotations

Q: What is the matrix for a rotation of angle ϕ about the z-axis?

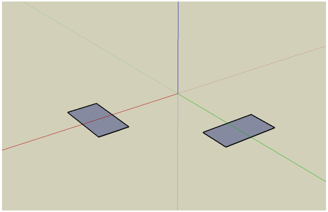


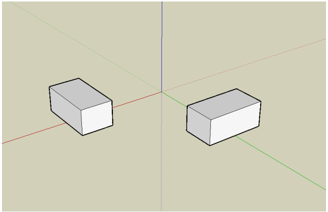
Rotation in 2D ($x - y$-plane):

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Transformations in 3D: rotations

Q: What is the matrix for a rotation of angle ϕ about the z-axis?





Rotation in 2D ($x-y$-plane):

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Rotation in 3D around z-axis:

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transformations in 3D: rotations

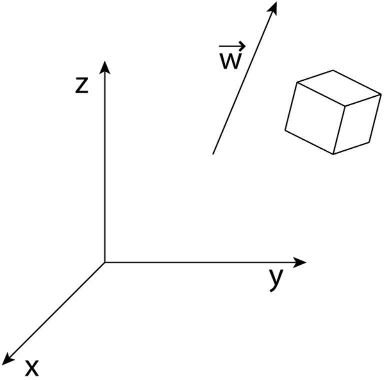
Q: What is the matrix for a rotation of angle ϕ about the directed line $(0,1,2)+t(3,0,4) \quad t>0$?

Transformations in 3D: rotations

We need a 3D transformation that rotates around an arbitrary vector $\vec{w}$.
How can we do that?

Idea:

1. Rotate vector $\vec{w}$ to z-axis
2. Do 3D rotation around z-axis
3. Rotate everything back to original position



Transformations in 3D: rotations

We already know how to rotate around the z-axis:

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transformations in 3D: rotations

But how do we get the other 2 rotation matrices?

$$\begin{pmatrix} & & \\ & ? & \\ & & \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} & & \\ & ? & \\ & & \end{pmatrix}$$

Rotating everything back to it's original position after we have done the rotation is easy.

Remember that the columns of the rotation matrix represent the images of the carthesian basis vectors under the transformation!

$$\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} & & \\ & ? & \\ & & \end{pmatrix}$$

Transformations in 3D: rotations

Now all we still need is the original rotation matrix

$$\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} & ? & \end{pmatrix}$$

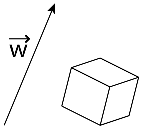
But that's just the inverse of the other rotation matrix.

Note: if we assume that $\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = 1$ we don't even need to calculate it, because the inverse of an orthogonal matrix is always its transposed!

$$\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{pmatrix}$$

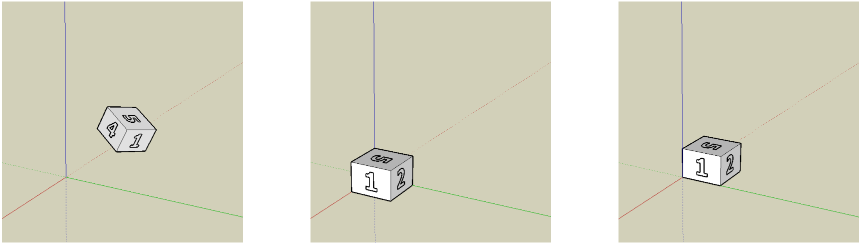
Transformations in 3D: rotations

Hmm, but we only have the vector $\vec{w}$. How do we get a whole coordinates system $\vec{u}$, $\vec{v}$, $\vec{w}$?

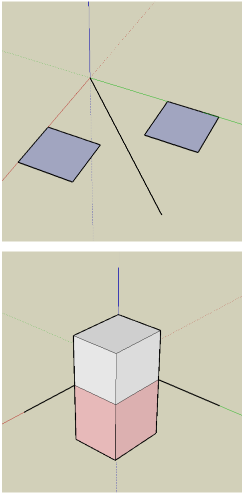


Transformations in 3D: rotations

Notice that such a rotation is not unique. (And it doesn't have to be as long as the two rotation matrices are correct.)



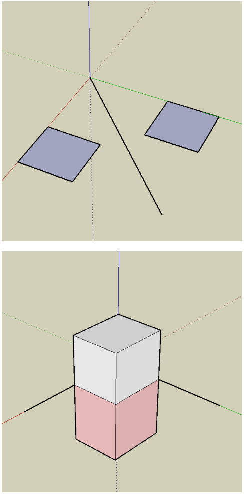
Transformations in 3D: reflections



Q: What is the matrix for reflection in XY-plane?

What happens to a random point (x, y, z) in 3D when we reflect it on the XY-plane?

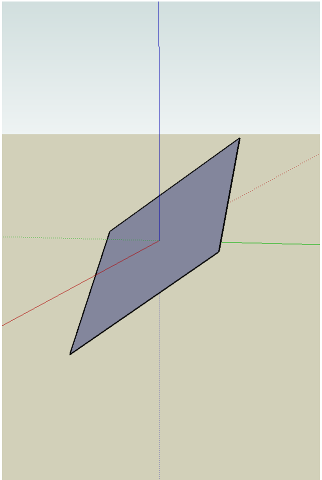
Transformations in 3D: reflections



Hence, the matrix for reflection in XY-plane is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

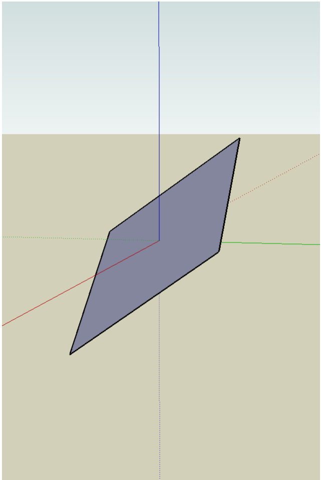
Transformations in 3D: reflections



Q: What is the matrix for reflection in XY-plane?

Q: What is the matrix for reflection in the plane $3x + 4y - 12z = 0$?

Transformations in 3D: reflections



Q: What is the matrix for reflection in XY-plane?

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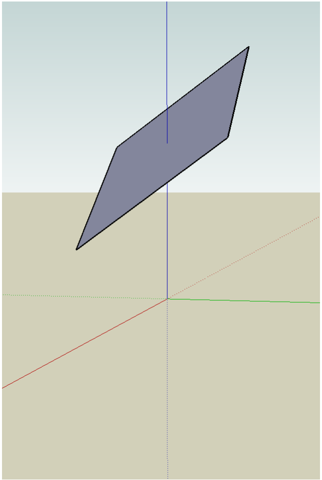
Transformations in 3D: reflections



Q: What is the matrix for reflection in XY-plane?

Q: What is the matrix for reflection in the plane $3x + 4y - 12z = 0$?

Q: What is the matrix for reflection in the plane $3x + 4y - 12z = 11$?