Graphics 2008/2009

T1

Midterm exam

Thu, Oct 02, 2008, 15:00–17:00

Comments and partial solutions

Note: these are not sample solutions, but rather sketches and comments which sometimes are more detailed than what we actually expected from you.

- **Do not open this exam until instructed to do so.**
- **Read the instructions on this page carefully.**

- You may write your answers in English, Dutch, or German.
  Use a pen, not a pencil. Avoid usage of the color red.
- You may not use books, notes, or any electronic equipment
  (including your cellphone, even if you just want to use it as a clock).
- Please put your student ID on the table so we can walk around and check it during the exam.
  You also have to show it to the instructor when you turn your exams in before leaving the room.
- Write down your name and student number on every paper you want to turn in.
  Additional paper is provided by us. You are not allowed to use your own paper.

- The exam should be doable in less than 1.5 hours. You have max. 2 hours to work on the questions.
  If you finish early, you may hand in your work and leave, except for the first half hour of the exam.
- The exam consists of 4 problems printed on 4 pages (including this one).
  It is your responsibility to check if you have a complete printout.
  If you have the impression that anything is missing, let us know.
- The maximum number of points you can score is 20.
  You need at least 18 points to get the best possible grade.

Good luck!

Problem 1: Vectors

Subproblem 1.1 [2 pt] Assume two vectors $\mathbf{a} = (3,8)^T$ and $\mathbf{b} = (6,16)^T$.

(a) Which of the following statements are correct?
(shortly explain your answer, multiple answers may apply)

1. **a** and **b** are parallel.
2. **a** and **b** are linearly independent.
3. **a** and **b** form a 2D basis.
4. **a** is a scalar multiple of **b**.

(b) Which of the statements 1. - 4. are correct if $\mathbf{a} = (8,3)^T$?
(shortly explain your answer, multiple answers may apply)

Solution/comments. *We know that two vectors are parallel if and only if one is a scalar multiple of the other. We also know that they are called linearly independent if they are not parallel and that two linearly independent vectors form a 2D basis. Therefore, for any pair of vectors, either statement 1. and 4. or statement 2. and 3. can be correct (exclusive or!).*

*In subproblem (a), vector **b** is a scalar multiple of vector **a**. Hence, statement 1. and 4. are correct. In subproblem (b), the vectors are no scalar multiple of each other. Hence, statement 2. and 3. are correct.*

Subproblem 1.2 [2 pt]

(a) What do we know about two random vectors **v** and **w** if their scalar product is zero, i.e. if $\mathbf{v} \cdot \mathbf{w} = 0$? Shortly explain your answer. (Note: think of *all* possible options.)

(b) What do we know about the value of the scalar product of two unit vectors **v** and **w** if the angle ϕ between them is between zero and 90°, i.e. if $0 < \phi < 90°$? Shortly explain your answer.

Solution/comments. *If you remember that* $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\ ||\mathbf{w}|| \cos\phi$*, then solving these 2 subproblems is easy. If the dot product is 0, either* $||\mathbf{v}||$ *or* $||\mathbf{w}||$ *must be zero (i.e. at least one of them is the null vector) or* $\cos\phi$ *must be zero (i.e.* ϕ *must be 90 degree, i.e. the vectors must be perpendicular).*

If they are unit vectors, their length is 1. Therefore, the scalar product is just the value of the cosine which for $0 < \phi < 90°$ *is* $0 < \mathbf{v} \cdot \mathbf{w} < 1$.

Problem 2: Basic geometric entities

Subproblem 2.1 [5 pt]

(a) What is the general form of a parametric equation of a plane in 3D? Write it down and explain the geometric interpretation of its components.

Solution/comments. *We can write a parametric equation either as*

$$\mathbf{p}(s,t) = \mathbf{p_0} + s\mathbf{v_1} + t\mathbf{v_2}$$

or as

$$\mathbf{p}(s,t) = \mathbf{p_0} + s(\mathbf{p_1} - \mathbf{p_0}) + t(\mathbf{p_2} - \mathbf{p_0}).$$

In the second case, $\mathbf{p_1}$ *and* $\mathbf{p_2}$ *are two points on the plane. In the first case,* $\mathbf{v_1}$ *and* $\mathbf{v_2}$ *are two vectors on the plane. They are sometimes called* direction vectors *because they specify the direction or orientation of the plane in the n-dim space. In both cases,* $\mathbf{p_0}$ *is a point on the plane, often called* support vector *because it describes the location of the plane in relation to the origin.*

s and t are the free parameters or variables used to get all the points on the plane. As characteristic for parametric equations, you see that we need n-1 parameters to describe an object in an n-dimensional space (here: n=3).

(Note: It was not necessary to use the terms direction and support vector as long as you described their purpose correctly. The comment on n-1 parameters in an n-dim space was also not needed in order to get full points.)

(b) Assume the following three points in $\mathbb{R}^3$:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Calculate a normal vector **n** for the plane defined by **a**, **b**, and **c**

Solution/comments. *We can get the normal vector by taking the cross product of any two vectors on the plane. Because we end up getting lots of 1's and 0's (which make calculation easier), it's a good idea to take vectors* $(\mathbf{b} - \mathbf{a})$ *and* $(\mathbf{c} - \mathbf{a})$*. This gives us*

$$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = (2,1,0) \times (0,1,2) = (2,-4,2).$$

Taking any other vectors on the plane than $(\mathbf{b} - \mathbf{a})$ *and* $(\mathbf{c} - \mathbf{a})$ *will result in a vector that is a scalar multiple of (2, -4, 2) and therefore also a normal vector (i.e. a correct solution).*

(c) What is the general form of an implicit representation of a plane in 3D? Write it down and explain the geometric interpretation of its components.

Solution/comments. *The implicit representation of a line is either*

$$f(x,y,z) = ax + by + cz + d = 0$$

or (in vector notation)

$$f(\mathbf{p}) = \mathbf{n}(\mathbf{p} - \mathbf{p_0}) = 0.$$

In contrast to the parametric equation (cf. (a)), we don't have two direction vectors but a normal vector **n** *(or (a, b, c)) of the plane that describes its orientation. In vector notation, we have a point* $\mathbf{p_0}$ *on the plane that describes its location in relation to the origin (similar to the support vector in a parametric equation). In the other form, the parameter d describes the distance of the plane from the origin. All points with coordinate values x,y, and z that fulfill the equation (i.e. =0) are on the plane.*

(d) Create an implicit representation of the plane defined by the three vectors **a**, **b**, and **c** given in (b). Do not just write down the solution but explain it (e.g. by using references to (c)).

Solution/comments. *Since we already calculated a normal vector and have three points on the plane, it is easiest to take the vector representation and just fill in the values. If we use, for example,* $\mathbf{p_0}$*, that gives us:*

$$(2,-4,2)(\mathbf{p}-(1,1,1)) = 0$$

If we use the other form, we need to calculate d by plugging in one of the three points into the equation. If you do this, you will end up with d=0 which means that the plane goes through the origin (note: it was not necessary to mention that).

Subproblem 2.2 [1 pt]

Assume you are calculating the intersection of a line and a plane in 3D. What are the possible number of solutions you can get (note: write down *all* possibilities)? What is the geometric interpretation of each case?

Solution/comments. *There are three options: In case the line is parallel to the plane, they don't intersect, i.e. we have 0 solutions. If the line is on the plane, the intersection is the line itself, i.e. we have an infinite number of solutions. In all other cases, the line intersects with the plane in a point, i.e. we have exactly one solution.*

Problem 3: Matrices

Subproblem 3.1 [1.5 pt] Which of the following answers is correct if s is a scalar value, $\mathbf{v}$ is a vector in $\mathbb{R}^3$, and $\mathbf{A}$ is a 2x3 matrix? (no explanation required)

(a) s**A** is (a1) a scalar, (a2) a vector in $\mathbb{R}^2$, (a3) a vector in $\mathbb{R}^3$, (a4) a 2x3 matrix, (a5) a 3x3 matrix, or (a6) undefined?

(b) Av is (b1) a scalar, (b2) a vector in $\mathbb{R}^2$, (b3) a vector in $\mathbb{R}^3$, (b4) a 2x3 matrix, (b5) a 3x3 matrix, or (b6) undefined?

(c) vA is (c1) a scalar, (c2) a vector in $\mathbb{R}^2$, (c3) a vector in $\mathbb{R}^3$, (c4) a 2x3 matrix, (c5) a 3x3 matrix, or (c6) undefined?

Solution/comments. *We multiply a scala with a matrix by multiplying each of it's entries. Hence, answer (a4) is correct. An n-dimensional vector can be seen as a nx1 matrix. Since we get a 2x1 matrix when we multiply a 2x3 matrix with a 3x1 matrix, answer (b2) is correct. Because the number of columns in the first factor needs to match the number of lines in the second one in order to do matrix multiplication, answer (c6) is obviously correct.*

(Note: It was not needed to give an explanation.)

Subproblem 3.2 [3.5 pt]

Assume we have three planes P_1, P_2, and P_3 which are defined by the following equations:

- $P_1 : 2x+2y+4z = 18$
- $P_2 : 1x+3y+4z = 19$
- $P_3 : 1x+2y+5z = 20$

(a) Write this down as a system of linear equations in matrix notation and solve using Gaussian elimination.

(b) What is the geometric interpretation of your solution?

(c) Assume we are replacing plane P_1 with a new plane P_1',

- $P_1' : 2x+4y+10z = 40$

Can the related system of linear equations still be solved? Explain why or why not. What is the geometric interpretation of this?

Solution/comments. *If you apply the rules from Gaussian elimination, you will realize that you always have to divide one line by 2 and then multiply it with -1 and add it to the other two lines. If you do this three times and make no arithmetic error, you will end up with the solution x=1, y=2, z=3. The solution is a point in 3D. Since it fulfills each of the equations, it must be the intersection point of the three planes.*

If we replace the first plane with P_1' and apply Gaussian elimination, we will end up with one line 0x + 0y + 0z = 0, because the last equation is just a scalar multiple of the first one. Geometrically this means that the equations present two identical planes. Hence, we have the itersection of two planes which is a line. Since we have three parameters with only two equations in our linear equation system, we can solve for one parameter and express the other ones in relation to it, thus getting indeed the equation of a line.

Problem 4: Transformations

Subproblem 4.1 [1 pt]

In $\mathbb{R}^2$, the matrix $\begin{pmatrix} cos\,\alpha & -sin\,\alpha \\ sin\,\alpha & cos\,\alpha \end{pmatrix}$ defines a counterclockwise rotation about the angle α about the origin. Give a 3x3 transformation matrix for a similar rotation about the x-axis in $\mathbb{R}^3$.

Solution/comments. *If you remember the images I showed and what I said during the lecture when discussing rotation around the z-axis, you realize that rotation around the x-axis does not change the x-values and for the y- and z-values it is basically a rotation in the y-z-plane. Hence, we can just take the 2d rotation matrix for the y- and z-coordinates and fill in some values for the x-coordinates that don't change it, i.e.*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\,\alpha & -sin\,\alpha \\ 0 & sin\,\alpha & cos\,\alpha \end{pmatrix}$$

Subproblem 4.2 [2.5 pt]

(a) Describe in your own words what happens to a point **p** in $\mathbb{R}^3$ if you apply the following transformation matrix to it:

$$\begin{pmatrix} 3 & 0 & 0 & x_m \\ 0 & 3 & 0 & y_m \\ 0 & 0 & 3 & z_m \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

How are the values in the last row of this matrix called and why do we need them?

(b) What happens if you apply the matrix from (a) to a vector **v** in $\mathbb{R}^3$. Explain, why your answer differs from the one given in (a).

Solution/comments. *The upper left 3x3 submatrix scales the vector (representing the point) by a factor of 3. The first three values in the right column represent a translation of the point to the position x_m, y_m, z_m. The values in the last row are called homogeneous coordinates. We need them because normal matrix multiplication does not allow us to do translations. However, by using them and setting the fourth value of the location vector to 1, we get exactly what we want, i.e. the x_m, y_m, and z_m values are added to the x, y, and z coordinates of the location vector but do not interfer with the rest of our calculations.*

Since vectors represent a direction but not a location, they should not be translated. We can achieve this by setting the value of the homogeneous coordinate of the vector to 0. If you multiply this with the given matrix, if is obvious, that the first three coordinate values are multiplied with a factor of three but anything else remains unchanged. So, the vector gets scaled by a factor of three but not translated. That's exactly what we want.

Subproblem 4.3 [1.5 pt]

Assume we have a vector **v** and a rotation matrix $\mathbf{M}_{rot}$, a scaling matrix $\mathbf{M}_{scale}$, and a reflection matrix $\mathbf{M}_{ref}$. Now, we first want to rotate the vector **v** using $\mathbf{M}_{rot}$. Then, we want to reflect the resulting vector using $\mathbf{M}_{ref}$. Finally, we want to scale this result using $\mathbf{M}_{scale}$.

(a) Write down the order in which we have to multiply the original vector with the three matrices to do this (i.e. replace *a*, *b*, *c* in $\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c\mathbf{v}$ with the correct indices *rot*, *scale*, and *ref*.)

(b) Why is the order important, i.e. why might we get a different result if we use a different order?

(c) What characteristic of matrix multiplication is the reason why we can replace the matrices $\mathbf{M}_a\mathbf{M}_b\mathbf{M}_c$ in the transformations above with a single matrix $\mathbf{M} = \mathbf{M}_a\mathbf{M}_b\mathbf{M}_c$?

Solution/comments. *Because matrix multiplication is not commutative, changing the orders of the matrices can also change the result. Hence, we have to multiply them to the vector from right to left what gives us $\boldsymbol{M}_{scale}\boldsymbol{M}_{ref}\boldsymbol{M}_{rot}$*

(Note: alternatively, you could have argued by the geometric interpretation of this matrix multiplication by saying that, for example, a reflection followed by a scaling does not necessarily lead to the same result as if we apply these transformations in reverse order.)

Since matrix multiplication is associative, $(\boldsymbol{M}_{scale}(\boldsymbol{M}_{ref}(\boldsymbol{M}_{rot}\ v)))$ and $(\boldsymbol{M}_{scale}(\boldsymbol{M}_{ref}(\boldsymbol{M}_{rot}))\ v)$ both will deliver the same result which is why we can replace the three matrices with a single matrix $\boldsymbol{M}$.