Homework Exam 3 2022-2023

My name and StudentID go here!

**Deadline:** 20 January 2023, 11:00

This homework exam has 5 questions for a total of 90 points. You can earn 10 additional points by a careful preparation of your hand-in: using a good layout, good spelling, good figures, no sloppy notation, no statements like “The algorithm runs in $n \log n$” (forgetting the $O(...)$ and forgetting to say that it concerns time), etc. Use lemmas, theorems, and figures where appropriate. Your final grade will be the number of points divided by 10. Unless stated otherwise, both randomized and deterministic solutions are allowed. In case you are asked to analyze the running time and your algorithm is randomized, analyze its expected running time.

**Question 1**

Let $P$ be a set of $n$ convex $k$-gons. You can assume that no three vertices of the $k$-gons are collinear, and that the polygons do not have horizontal edges.

a. **(10 points)** Consider a randomized incremental construction algorithm to compute the topmost point $p^*$ that lies in the intersection of all polygons in $P$. I.e. the algorithm shuffles the polygons in $P$ and adds them one by one while maintaining the highest point $p^*_j$ in $T^j_i = \cap_{i=1}^j P_i$ of the polygons $P_1, \ldots, P_j$ encountered so far.

Describe the two main remaining subroutines that you need in such an algorithm, and how to implement them. Furthermore, give/analyze the running times of these subroutines.

b. **(10 points)** Formally analyze the expected running time of the randomized incremental construction algorithm sketched above.

**Question 2** (15 points)

Let $S$ be a set of $n$ axis-parallel squares. Develop a data structure that can store $S$ that can answer the following queries: given a point $q$, report a largest square $S^* \in S$ that contains $q$. Argue/prove that your data structure answers queries correctly. Aim for the fastest queries possible, while using $O(n \log^c n)$ space (for some constant $c$).

The number of points awarded will depend on the query time and the space used by your data structure.

**Question 3** (15 points)

Let $T$ be a kD-tree on a set $P$ of $n$ points in $\mathbb{R}^2$, in which each node has been annotated with the number of points in its subtree. Analyze the worst case query time for a range counting query on $T$ with a query disk $D$. Argue that your analysis is tight in the worst case.

**Question 4**

Let $P$ be a set of $n$ points in $\mathbb{R}^2$, and let $NN(p)$ denote the (Euclidean) nearest-neighbor of $p$. That is, $NN(p) = \arg \min_{q \in P} \|pq\|$. 

a. **(10 points)** Prove that the Voronoi regions of $p$ and $NN(p)$ are adjacent in the Voronoi diagram $VD(P)$ of $P$.

b. **(10 points)** Describe an $O(n \log n)$ time algorithm to compute, for every point $p \in P$, its nearest neighbor $NN(p)$. 

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<table>
<thead>
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<th>Question 5 (20 points)</th>
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<td>Let $R$ be a set of $n$ “red” points in $\mathbb{R}^2$, let $B$ be a set of $n$ “blue” points in $\mathbb{R}^2$, let $\text{age} : R \cup B \to \mathbb{R}$ be a function that assigns an age to every point, and let $\Delta &gt; 0$ be some real number. You can again assume that all coordinates and ages are unique, and that there are no three colinear or four concyclic points. Design an algorithm to compute, for every red point $r \in R$, the closest (in terms of the Euclidean distance) blue point $b$ among $B$ for which $\text{age}(b) \in [\text{age}(r), \text{age}(r) + \Delta]$. Your algorithm should run in subquadratic time.</td>
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