Homework Exam 1 2022-2023

Model Solution

Deadline: 25 November 2022, 13:15

This homework exam has 1 question for a total of 9 points. You can earn an additional point by a careful preparation of your hand-in: using a good layout, good spelling, good figures, no sloppy notation, no statements like “The algorithm runs in \( n \log n \)” (forgetting the \( O(\cdot) \) and forgetting to say that it concerns time), etc. Use lemmas, theorems, and figures where appropriate.

**Question 1** (9 points)

Let \( r \in \mathbb{R}^2 \) be a “red” point, and let \( B \) be a set of \( n \) “blue” points in \( \mathbb{R}^2 \). You can assume that the points are in general position; meaning that no two points have the same \( x \)-coordinate or the same \( y \)-coordinate, and that no three points lie on a line. A triangle is “bichromatic” when its vertices are either red or blue, and it has at least one vertex of either color. Develop an \( O(n \log n) \) time algorithm to find a maximum area “bichromatic” triangle \( \Delta^* \) on \( \{r\}, B \).

**Hint:** You can use the following fact. A function \( f[1..n] \to \mathbb{R} \) is unimodal if (and only if) it has a single (local) maximum. A maximum of \( f \) can be computed in \( O(T \log n) \) time, where \( T \) is the time it takes to evaluate a single value \( f(i) \) with \( i \in [1..n] \). In particular, using the following function \( \text{TernarySearch}(\{a..b\}, f) \):

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function TernarySearch([a..b], f)
    n ← b − a
    if n < 3 then evaluate \( f(i) \) for each \( i \in [a..b] \) and return \( \max_i f(i) \)
    else
        \( m_1 ← a + \lfloor n/3 \rfloor \); \( m_2 ← a + \lceil 2n/3 \rceil \)
        if \( f(m_1) < f(m_2) \) then TernarySearch([\( m_1 .. b \)], f)
        else TernarySearch([\( a .. m_2 \)], f)
        end if
    end if
end function
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The key idea is in the following lemma, which proves that there exists an optimal, that is, maximum area, bichromatic triangle \( \Delta^* = \Delta(r, b, b') \) for which \( b \) and \( b' \) are vertices of the convex hull \( \mathcal{CH}(B) \) of \( B \). This restricts the number of candidate triangles. Moreover, it actually gives us a way to find \( b' \) efficiently when we are given point \( b \). This allows us to develop an \( O(n \log n) \) time algorithm.

**Lemma 1.** Let \( b \in B \) be a point that does not appear on \( \mathcal{CH}(B) \), and let \( H^+ \) be any halfplane whose bounding line goes through \( b \). Then \( H^+ \) contains a point from \( B \).

**Proof.** Assume, by contradiction, that \( b \) does not appear on \( \mathcal{CH}(B) \), yet \( H^+ \) is empty. Since \( b \) lies in the interior of \( \mathcal{CH}(B) \), it follows \( \mathcal{CH}(B) \) must intersect \( H^+ \) in some region \( R \) that has non-zero area. Now consider the set \( \mathcal{CH}(B) \cap H^- \) (where \( H^- = \mathbb{R}^2 \setminus H^+ \) is the other halfplane defined by the bounding line of \( H^+ \)). Since both \( \mathcal{CH}(B) \) and \( H^- \) are convex, so is \( \mathcal{CH}(B) \cap H^- \). Furthermore, since \( H^+ \) contains no points from \( B \), we have that \( B \subseteq \mathcal{CH}(B) \cap H^- \). Since \( R \) is has non-zero area, and thus \( \mathcal{CH}(B) \cap H^- \subseteq \mathcal{CH}(B) \). This contradicts the definition of \( \mathcal{CH}(B) \). This completes the proof.

**Lemma 2.** Let \( r \) and \( b \) be fixed, and let \( b' \) be a vertex of \( \mathcal{CH}(B) \) whose distance to the line \( \ell_{rb} \) through \( r \) and \( b \) is maximal. The triangle \( \Delta(r, b, b') \) has maximum area among all bichromatic triangles with vertices \( r \) and \( b \).

**Proof.** Assume, by contradiction, that \( \Delta(r, b, b') \) is a maximal area triangle, but that \( b' \) is not a vertex of \( \mathcal{CH}(B) \) for which the distance to the line \( \ell_{rb} \) is maximal, nor does there exist a triangle \( \Delta(r, b, b'') \) with larger area.
We first compute the convex hull $\text{CH}(B)$. Since (by assumption) $b'$ is not a vertex of the convex hull, Lemma 1 gives us that $H$ cannot be empty. So, let $b'' \in B$ be a point in $H$. Observe that the distance from a point $q \in \mathbb{R}^2$ to line $\ell_{rb}$ is the height of $\Delta(r, b, q)$. Since the distance from $b''$ to $\ell_{rb}$ is larger than the distance from $b'$ to $\ell_{rb}$, it thus follows that the height of $\Delta(r, b, b'')$ is larger than that of $\Delta(r, b, b')$. Since both triangles have the same base, the area of $\Delta(r, b, b'')$ is also larger than that of $\Delta(r, b, b')$. Contradiction.

**Lemma 3.** There exists an optimal triangle $\Delta^* = \Delta(r, b, b')$ for which (i) $b$ is a vertex of $\text{CH}(B)$, and (ii) $b'$ is a vertex of $\text{CH}(B)$ whose distance to the line $\ell_{rb}$ through $r$ and $b$ is maximal.

**Proof.** By applying Lemma 2 twice: We first fix $r$ and $b'$, and use the Lemma 2 to obtain that $b$ must be a vertex of $\text{CH}(B)$, thus establishing (i). We then apply Lemma 2 once more fixing points $r$ and $b$ to obtain (ii).

Fix a point $b \in \text{CH}(B)$, and let $H^+$ and $H^-$ denote the two halfspaces bounded by the line $rb$. Let $b = b_1, \ldots, b_k$ denote the points on $\text{CH}(B)$ in order along $\text{CH}(B)$ in $H^+$, and let $h(i)$ denote the distance from $b_i$ to the line $\ell_{rb}$.

**Lemma 4.** The function $h$ is unimodal.

**Proof.** For ease of description, rotate and translate the plane so that $H^+$ is bounded from below by the $x$-axis (so $r$ and $b$ lie on the $x$-axis). This way, $h(i)$ is simply the $y$-coordinate of $b_i$. We now argue that $h$ is unimodal.

Assume, by contradiction, that $h$ has two local maxima at $i$ and $j$, with $i < j - 1$. Let $m \in \{i + 1, j - 1\}$ minimize $h$ (among $i + 1$ and $j - 1$). Hence, the $y$-coordinate of $b_m$ is smaller than that of $b_i$ and $b_j$. It then follows $b_m$ lies strictly below the oriented line through $b_i$ and $b_j$. However, therefore $b_m$ cannot lie on $\text{CH}(B)$ (or at least on the portion of $\text{CH}(B)$ in between $b_i$ and $b_j$). Contradiction.

Analogous to Lemma 4 we the distance from the vertices of $\text{CH}(B) \cap H^-$ is unimodal. This, together with Lemma 2 then suggests an $O(\log n)$ time algorithm to find a triangle $\Delta(r, b, b')$ with maximal area among all bichromatic triangles with vertices $r$ and $b$ (provided we have access to $\text{CH}(B)$):

- Using a binary search, we find the last vertex $b_k$ such that $b = b_1, \ldots, b_k$ lie in $H^+$.
- Using the algorithm TERNARYSEARCH([1..k], h), we then find a vertex $b_i \in b_1, \ldots, b_k$ with maximum distance to $\ell_{rb}$. The triangle $\Delta(r, b, b_i)$ has maximum area among $b_1, \ldots, b_k$.
- We repeat the previous step for the other points on $\text{CH}(B)$ (that are in $H^-$).

It now follows that we have an $O(n \log n)$ time algorithm to compute an maximal area bichromatic triangle. We first compute the convex hull $\text{CH}(B)$, and then use the above approach for each vertex $b \in \text{CH}(B)$. We report the triangle $\Delta(r, b, b')$ with maximal area that we find. Correctness follows from Lemmas 3 and 4.

Computing the convex hull takes $O(n \log n)$ time. The above approach takes $O(\log n)$ time per candidate point $b$, and there are $O(n)$ candidate points $b$. We thus obtain the following result.

**Theorem 5.** Given a point $r \in \mathbb{R}^2$, and a set $B$ of $n$ points in $\mathbb{R}^2$, we can compute a maximal area bichromatic triangle $\Delta(r, b, b')$ on $r, B$ in $O(n \log n)$ time.