Computational Geometry

Lecture 6: Smallest enclosing circles and more
Facility location

Given a set of houses and farms in an isolated area. Can we place a helicopter ambulance post so that each house and farm can be reached within 15 minutes?

Where should we place an antenna so that a number of locations have maximum reception?
Facility location in geometric terms

Given a set of points in the plane. Is there any point that is within a certain distance of these points?

Where do we place a point that minimizes the maximum distance to a set of points?
Given a set of points in the plane, compute the smallest enclosing circle
Observation: It must pass through some points, or else it cannot be smallest

- Take any circle that encloses the points, and reduce its radius until it contains a point \( p \)
- Move center towards \( p \) while reducing the radius further, until the circle contains another point \( q \)
Smallest enclosing circle

- Move center on the bisector of $p$ and $q$ towards their midpoint, until:
  (i) the circle contains a third point, or
  (ii) the center reaches the midpoint of $p$ and $q$
**Question:** Does the “algorithm” of the previous slide work?
Observe: A smallest enclosing circle has (at least) three points on its boundary, or only two in which case they are diametrally opposite.

Question: What is the extra property when there are three points on the boundary?
Construction by randomized incremental construction

*incremental construction*: Add points one by one and maintain the solution so far

*randomized*: Use a random order to add the points
Adding a point

Let $p_1, \ldots, p_n$ be the points in random order

Let $C_i$ be the smallest enclosing circle for $p_1, \ldots, p_i$

Suppose we know $C_{i-1}$ and we want to add $p_i$

- If $p_i$ is inside $C_{i-1}$, then $C_i = C_{i-1}$
- If $p_i$ is outside $C_{i-1}$, then $C_i$ will have $p_i$ on its boundary
Adding a point

\[ C_{i-1} \]

\[ p_i \]

\[ C_i \]
**Question:** Suppose we remembered not only $C_{i-1}$, but also the two or three points defining it. It looks like if $p_i$ is outside $C_{i-1}$, the new circle $C_i$ is defined by $p_i$ and some points that defined $C_{i-1}$. Why is this false?
Adding a point
How do we find the smallest enclosing circle of $p_1, \ldots, p_{i-1}$ with $p_i$ on the boundary?

We study the *new(!)* geometric problem of computing the smallest enclosing circle with a given point $p$ on its boundary.
Smallest enclosing circle with point

Given a set \( P \) of points and one special point \( p \), determine the smallest enclosing circle of \( P \) that must have \( p \) on the boundary

**Question:** How do we solve it?
Randomized incremental construction

Construction by **randomized incremental construction**

*incremental construction*: Add points one by one and maintain the solution so far

*randomized*: Use a random order to add the points
Adding a point

Let $p_1, \ldots, p_{i-1}$ be the points in random order

Let $C'_j$ be the smallest enclosing circle for $p_1, \ldots, p_j$ ($j \leq i-1$) and with $p$ on the boundary

Suppose we know $C'_{j-1}$ and we want to add $p_j$

- If $p_j$ is inside $C'_{j-1}$, then $C_j = C'_{j-1}$
- If $p_j$ is outside $C'_{j-1}$, then $C'_j$ will have $p_j$ on its boundary (and also $p$ of course!)
Adding a point

Adding a point $p_j$ to the smallest enclosing circle $C'_{j-1}$.
How do we find the smallest enclosing circle of $p_1, \ldots, p_{j-1}$ with $p$ and $p_j$ on the boundary?

We study the new(!) geometric problem of computing the smallest enclosing circle with two given points on its boundary.
Given a set $P$ of points and two special points $p$ and $q$, determine the smallest enclosing circle of $P$ that must have $p$ and $q$ on the boundary.

**Question:** How do we solve it?
Two points known
Two points known

\[ p \quad q \]
Assume w.l.o.g. that $p$ and $q$ lie on a vertical line. Let $\ell$ be the line through $p$ and $q$ and let $\ell'$ be their bisector.

For all points left of $\ell$, find the one that, together with $p$ and $q$, defines a circle whose center is leftmost $\rightarrow p_l$.

For all points right of $\ell$, find the one that, together with $p$ and $q$, defines a circle whose center is rightmost $\rightarrow p_r$.

Decide if $C(p,q,p_l)$ or $C(p,q,p_r)$ or $C(p,q)$ is the smallest enclosing circle.
Two points known

\[ C(p, q, p_r) \]

\[ C(p, q, p_l) \]
Analysis: two points known

Smallest enclosing circle for $n$ points with two points already known takes $O(n)$ time, worst case
Algorithm: one point known

- Use a random order for \( p_1, \ldots, p_n \); start with \( C_1 = C(p, p_1) \)
- for \( j \leftarrow 2 \) to \( n \) do
  - If \( p_j \) in or on \( C_{j-1} \) then \( C_j = C_{j-1} \); otherwise, solve smallest enclosing circle for \( p_1, \ldots, p_{j-1} \) with two points known (\( p \) and \( p_j \))
Analysis: one point known

If only one point is known, we used randomized incremental construction, so we need an *expected time analysis*.
Analysis: one point known

**Backwards analysis**: Consider the situation *after* adding $p_j$, so we have computed $C_j$.
Analysis: one point known

The probability that the $j$-th addition was expensive is the same as the probability that the smallest enclosing circle changes (decreases in size) if we remove a random point from the $j$ points.
Analysis: one point known

This probability is $2/j$ in the left situation and $1/j$ in the right situation.
Analysis: one point known

The expected time for the $j$-th addition of a point is

$$\frac{j-2}{j} \cdot \Theta(1) + \frac{2}{j} \cdot \Theta(j) = O(1)$$

or

$$\frac{j-1}{j} \cdot \Theta(1) + \frac{1}{j} \cdot \Theta(j) = O(1)$$

The expected running time of the algorithm for $n$ points is:

$$\Theta(n) + \sum_{j=2}^{n} \Theta(1) = \Theta(n)$$
Analysis: one point known

Smallest enclosing circle for $n$ points with one point already known takes $\Theta(n)$ time, expected
Algorithm: smallest enclosing circle

- Use a random order for $p_1, \ldots, p_n$; start with $C_2 = C(p_1, p_2)$
- for $i \leftarrow 3$ to $n$ do
  If $p_i$ in or on $C_{i-1}$ then $C_i = C_{i-1}$; otherwise, solve smallest enclosing circle for $p_1, \ldots, p_{i-1}$ with one point known ($p_i$)
Analysis: smallest enclosing circle

For smallest enclosing circle, we used randomized incremental construction, so we need an *expected time analysis*.
Backwards analysis: Consider the situation after adding $p_i$, so we have computed $C_i$.
The probability that the $i$-th addition was expensive is the same as the probability that the smallest enclosing circle changes (decreases in size) if we remove a random point from the $i$ points.
This probability is $\frac{3}{i}$ in the left situation and $\frac{2}{i}$ in the right situation.
Analysis: smallest enclosing circle

The expected time for the $i$-th addition of a point is

$$\frac{i - 3}{i} \cdot \Theta(1) + \frac{3}{i} \cdot \Theta(i) = O(1)$$

or

$$\frac{i - 2}{i} \cdot \Theta(1) + \frac{2}{i} \cdot \Theta(i) = O(1)$$

The expected running time of the algorithm for $n$ points is:

$$\Theta(n) + \sum_{i=3}^{n} \Theta(1) = \Theta(n)$$
Result: smallest enclosing circle

**Theorem** The smallest enclosing circle for $n$ points in plane can be computed in $O(n)$ expected time
Randomized incremental construction algorithms of this sort (compute an ‘optimal’ thing) work if:

- The test whether the next input object violates the current optimum must be possible and fast
- If the next input object violates the current optimum, finding the new optimum must be an *easier* problem than the general problem
- The thing must already be defined by $O(1)$ of the input objects
- Ultimately: the analysis must work out
**Width**: Given a set of $n$ points in the plane, compute the smallest distance between two parallel lines that contain the points (narrowest strip)
**Width**: Given a set of \( n \) points in the plane, compute the smallest distance between two parallel lines that contain the points (narrowest strip)

**Theorem**: The width of a set of \( n \) points can be computed in \( O(n \log n) \) time.
**Property:** The width is always determined by three points of the set

**Idea:** Maintain the two lines defining the width to have a fast test for violation.
Question: How about adding a point? If the new point lies inside the narrowest strip we are fine, but what if it lies outside?
Adding a point
Adding a point
A good reason to be very suspicious of randomized incremental construction as a working approach is *non-uniqueness* of a solution.
**Question:** Can we compute the minimum axis-parallel bounding box by randomized incremental construction?
Yes, in $O(n)$ expected time

... but a normal incremental algorithm does it in $O(n)$ worst case time
**Problem 1:** Given $n$ disks in the plane, can we compute the lowest point in their common intersection efficiently by randomized incremental construction?

**Problem 2:** Given $n$ disks in the plane, can we compute the lowest point in their union efficiently by randomized incremental construction?
**Problem:** Given a set of $n$ red and blue points in the plane, can we decide efficiently if they have a separating line?
One-guardable polygons

**Problem:** Given a simple polygon with $n$ vertices, can we decide efficiently if one guard is enough?
One-guardable polygons

It can easily happen that a problem is an instance of linear programming.

Then don’t devise a new algorithm, just explain how to transform it, and show that it is correct (that your problem is really solved that way).