Elementary Maths for GMT - 2016 - Homework Exam 2

This homework exam consists of 4 questions. You can earn a total of 100 points. You may discuss questions and approaches on a high level with your fellow students, but must write down your own solutions. The first 10 points can be obtained by following these rules:

- Hand in your solutions on paper.
- Write clearly and legibly, or use a type setting system (such as \LaTeX).
- Write your name and student number on each sheet you hand in, and use page numbers.

Be concise in your answers. An explanation of how you obtained your answer is not mandatory unless specified in the question; however, a short derivation may be helpful in case of arithmetic mistakes.

**Question 1** [20 points]
Let \( f : (0, \infty) \to \mathbb{R} \) be defined as \( f(x) = e^{-x} \), and \( g : (0, \infty) \to \mathbb{R} \) be defined as \( g(x) = 1/x \).

(a) Is \( f \) a probability density function?
(b) Is \( g \) a probability density function?
(c) Is \( g \in \mathcal{O}(f) \)?
(d) Is \( f \in \mathcal{O}(g) \)?

**Question 2** [20 points]
I have six students in my class: one Chinese, two Greek, and three Dutch. For an assignment, I pair up the students in order: the Chinese and first Greek students form a pair, the second Greek and first Dutch student form a pair, and the remaining two Dutch students form a pair.

(a) I randomly select one of my students, and I observe that he or she is Dutch. Compute the probability that the partner of this student is Greek.

For a second assignment, the students are randomly paired up into three pairs of two students.

(b) I randomly select one of my students, and I observe that he or she is Dutch. Compute the probability that the partner of this student is Greek.

**Question 3** [25 points]
Let \( G = (V, E) \) be a connected graph. For a given vertex \( x \in V \), I define \( N(x) \) to be the set of neighbours of \( x \) in \( G \): \( N(x) = \{ y \in V : (x, y) \in E \} \). Given a set of vertices \( X \subseteq V \), I define \( N(X) \) to be the set of all neighbours of elements in \( X \): \( N(X) = \bigcup_{x \in X} N(x) \).

Now consider the following algorithm, which computes the length of the shortest path between two vertices of \( G \).

\[
\begin{align*}
\text{function} & \quad \text{myAwesomeShortestPathAlgorithm} \\
\text{input:} & \quad G = (V, E): \text{a connected graph}, \ s, t \in V: \text{two special vertices} \\
S & = \{s\}; \\
T & = V \setminus \{s\}; \\
d & = 0; \\
\text{while} & \quad t \notin S \quad \text{do} \\
\quad H & = N(S) \cap T; \\
\quad S & = S \cup H; \\
\quad T & = T \setminus H; \\
\quad d & = d + 1; \\
\text{end} \\
\text{return} & \quad d;
\end{align*}
\]

(a) Does this algorithm work? Argue why.
(b) Analyse the worst-case running time of this algorithm.
(c) What would you change to make the algorithm more efficient?
Question 4

In four-dimensional space, two linearly independent planes (two-dimensional subspaces) always intersect in a single point. Consider the planes $A$ and $B$, where $A$ goes through the points $a_1 = (2, 5, 0, 0)$, $a_2 = (3, 6, 0, 1)$ and $a_3 = (2, 7, 0, -1)$, and where $B$ goes through the points $b_1 = (0, -1, 0, 1)$, $b_2 = (3, -2, 3, 3)$ and $b_3 = (2, -1, 2, 3)$.

(a) Express the intersection of $A$ and $B$ as a matrix equation.

(b) Compute the intersection of $A$ and $B$. 