Elementary Maths for GMT - 2016 - Homework Exam 2
SOLUTIONS

This homework exam consists of 4 questions. You can earn a total of 100 points. You may discuss questions and approaches on a high level with your fellow students, but must write down your own solutions. The first 10 points can be obtained by following these rules:

- Hand in your solutions on paper.
- Write clearly and legibly, or use a type setting system (such as L\LaTeX).
- Write your name and student number on each sheet you hand in, and use page numbers.

Be concise in your answers. An explanation of how you obtained your answer is not mandatory unless specified in the question; however, a short derivation may be helpful in case of arithmetic mistakes.

Question 1 [20 points]
Let \( f : (0, \infty) \to \mathbb{R} \) be defined as \( f(x) = e^{-x} \), and \( g : (0, \infty) \to \mathbb{R} \) be defined as \( g(x) = 1/x \).

(a) Is \( f \) a probability density function?
(b) Is \( g \) a probability density function?
(c) Is \( g \in O(f) \)?
(d) Is \( f \in O(g) \)?

Solution

(a) Yes. The value of \( f \) is always positive, and the integral \( \int_{0}^{\infty} f(x)dx = 1 \).
(b) No. The integral \( \int_{0}^{\infty} g(x)dx = \infty \) (does not converge).
(c) No. \( \forall a, c : \exists x > a : g(x) > cf(x) \). For instance, take any \( x > \max\{2, a, c\} \).
(d) Yes. \( \exists a, c : \forall x > a : f(x) < cg(x) \). For instance, take \( a = c = 1 \).

Question 2 [20 points]
I have six students in my class: one Chinese, two Greek, and three Dutch. For an assignment, I pair up the students in order: the Chinese and first Greek students form a pair, the second Greek and first Dutch student form a pair, and the remaining two Dutch students form a pair.

(a) I randomly select one of my students, and I observe that he or she is Dutch. Compute the probability that the partner of this student is Greek.
(b) I randomly select one of my students, and I observe that he or she is Dutch. Compute the probability that the partner of this student is Greek.

Solution

(a) I observed a Dutch student. There are 3 Dutch students, two whom have a Dutch partner (each other), and one of whom has a Greek partner. So, the probability that their partner is Greek is \( \frac{1}{3} \).

We can also prove this formally. Let \( X : \Omega \to \{C, G, D\} \) be the nationality of the first student we pick, and let \( Y : \Omega \to \{C, G, D\} \) be the nationality of the second student we pick. Then a priori, \( P(X = C) = P(Y = C) = \frac{1}{6} \), \( P(X = G) = P(Y = G) = \frac{1}{3} \), and \( P(X = D) = P(Y = D) = \frac{1}{2} \). However, \( X \) and \( Y \) are not independent: there are only six possible combined values, each with probability \( \frac{1}{6} \): \( CG, GC, GD, DG, DD, \) and \( DD\).

We are interested in the probability \( P(Y = G \mid X = D) \). We use Bayes’ rule to find \( P(Y = G \mid X = D) = P(Y = G \land X = D)/P(X = D) = \frac{1}{6}/\frac{1}{2} = \frac{1}{3} \).
(b) Since the students are random, the partner of the observed student is chosen uniformly at random from the remaining 5 students. Of those students, 2 are Greek. So the probability is $\frac{2}{5}$.

We can prove this by considering all possible compositions of the student pairs. There are three possible pairings: $CG - GD - DD$, $CD - GD - GD$, and $CD - GG - DD$. Their probabilities are: $\frac{2}{5}$, $\frac{2}{5}$, and $\frac{1}{5}$ (we can find this by listing all possible permutations of the six students and counting how many fall in each category, or by multiplying the probabilities directly). In the first case, the probability is $\frac{1}{3}$, as computed in the previous question. In the second case, the probability is $\frac{2}{3}$, since here two of the three Dutch students have a Greek partner (we can also compute this formally, similar to the previous question). In the last case, the probability is 0.

Therefore, the total probability is $\frac{2}{5} \cdot \frac{1}{3} + \frac{2}{5} \cdot \frac{2}{3} + \frac{1}{5} \cdot 0 = \frac{2}{5}$.

**Question 3**

Let $G = (V, E)$ be a connected graph. For a given vertex $x \in V$, I define $N(x)$ to be the set of neighbours of $x$ in $G$: $N(x) = \{y \in V : (x, y) \in E\}$. Given a set of vertices $X \subseteq V$, I define $N(X)$ to be the set of all neighbours of elements in $X$: $N(X) = \bigcup_{x \in X} N(x)$.

Now consider the following algorithm, which computes the length of the shortest path between two vertices of $G$.

```python
function myAwesomeShortestPathAlgorithm
    input: $G = (V, E)$: a connected graph, $s, t \in V$: two special vertices
    $S = \{s\}$; $T = V \setminus \{s\}$; $d = 0$;
    while $t \notin S$
do
        $H = N(S) \cap T$;
        $S = S \cup H$;
        $T = T \setminus H$;
        $d = d + 1$;
    end
    return $d$;

(a) Does this algorithm work? Argue why.

(b) Analyse the worst-case running time of this algorithm.

(c) What would you change to make the algorithm more efficient?

**Solution**

(a) Yes, the algorithm works. At each iteration of the while loop, it discovers all nodes at distance $d$ from $s$, incrementing $d$ each time. The nodes at distance $d + 1$ from $s$ are exactly all nodes that are neighbouring to a node at distance $d$ (or less) from $s$, but which itself is not a node of distance $d$ or less.

(b) The algorithm has a while loop which is executed at most $|V|$ times, since after each step, the cardinality of $S$ increases and once $S$ contains all vertices, the algorithm terminates. Inside the while loop, it repeatedly computes all neighbours of $S$. The number of neighbours of $S$ is $O(|V|)$ in the worst case, leading to a $O(|V|^2)$ algorithm in the worst case.

Note that this analysis assumes constant-time set membership tests. If we evaluate $N$ by explicitly iterating over the incident edges, the algorithm takes even longer.

(c) We can improve the algorithm by not computing the neighbours of all vertices in $S$ each time, but only of those vertices at distance exactly $d$. That is, we can exchange the line $H = N(S) \cap T$ by $H = N(H) \cap T$. The algorithm now runs in linear time.
Question 4

In four-dimensional space, two linearly independent planes (two-dimensional subspaces) always intersect in a single point. Consider the planes $A$ and $B$, where $A$ goes through the points $a_1 = (2, 5, 0, 0)$, $a_2 = (3, 6, 0, 1)$ and $a_3 = (2, 7, 0, -1)$, and where $B$ goes through the points $b_1 = (0, -1, 0, 1)$, $b_2 = (3, -2, 3, 3)$ and $b_3 = (2, -1, 2, 3)$.

(a) Express the intersection of $A$ and $B$ as a matrix equation.

(b) Compute the intersection of $A$ and $B$.

Solution

(a) Let $\vec{u}_1 = a_2 - a_1 = (1, 1, 0, 1)$ and $\vec{u}_2 = a_3 - a_1 = (0, 2, 0, -1)$. Let $\vec{v}_1 = b_2 - b_1 = (3, -1, 3, 2)$ and $\vec{v}_2 = b_3 - b_1 = (2, 0, 2, 2)$.

Now we need to solve $p = a_1 + \lambda \vec{u}_1 + \mu \vec{u}_2 = b_1 + \theta \vec{v}_1 + \phi \vec{v}_2$.

Consider the matrix

$$M = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & 2 \\ 1 & -1 & 2 & 2 \end{pmatrix}$$

Then we can rewrite this as $M(\lambda, \mu, -\theta, -\phi) = b_1 - a_1 = (-2, -6, 0, 1)$, or in other words, $(\lambda, \mu, -\theta, -\phi) = M^{-1}(-2, -6, 0, 1)$.

(b) We find

$$M^{-1} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 1 & -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{3}{2} & \frac{1}{2} & 1 & 1 \end{pmatrix}$$

and thus $M^{-1}(-2, -6, 0, 1) = (-2, -\frac{7}{3}, -\frac{2}{3}, 1)$.

We find $p = (2, 5, 0, 0) - 2(1, 1, 0, 1) - \frac{7}{3}(0, 2, 0, -1) = (0, -\frac{5}{3}, 0, \frac{1}{3})$. 