Elementary Maths for GMT - 2016 - Homework Exam 1
SOLUTIONS

This homework exam consists of 3 questions. You can earn a total of 100 points. You may discuss questions and approaches on a high level with your fellow students, but must write down your own solutions. The first 10 points can be obtained by following these rules:
• Hand in your solutions on paper.
• Write clearly and legibly, or use a type setting system (such as \LaTeX).
• Write your name and student number on each sheet you hand in, and use page numbers.

Be concise in your answers. An explanation of how you obtained your answer is not mandatory unless specified in the question; however, a short derivation may be helpful in case of arithmetic mistakes.

Question 1 [20 points]
Let \( f : (0, 1) \times (0, 1) \to \mathbb{R} \) be defined as \( f(x, y) = \frac{x}{y} \).

(a) Is \( f \) injective, surjective, bijective, or none of these?
(b) Compute the gradient of \( f \).
(c) What is the limit of \( f \) at \((0, 0)\)?

Solution

(a) \( f \) is not injective, because there are different elements in the domain that map to the same number, for instance, \( f\left(\frac{1}{2}, \frac{1}{2}\right) = f\left(\frac{1}{3}, \frac{1}{3}\right) = 1 \). \( f \) is also not surjective, because there are numbers in \( \mathbb{R} \) that cannot be obtained by \( f \), for example, there are no \( x \) and \( y \) between 0 and 1 such that \( \frac{x}{y} = -1 \). Therefore, \( f \) is neither.

(b) The gradient of \( f \), \( \nabla f \), is the vector of partial derivatives, \( \nabla f(x, y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \). We find \( \frac{\partial f}{\partial x} = \frac{1}{y} \), and \( \frac{\partial f}{\partial y} = -\frac{x}{y^2} \), so, \( \nabla f(x, y) = (\frac{1}{y}, -\frac{x}{y^2}) \).

(c) The limit is undefined, because the value is different depending on the direction from which we approach the point \((0, 0)\). For instance, approaching along the line \( y = x \), we find the limit to be 1, but along the line \( y = 2x \), we find the limit to be \( \frac{1}{2} \). There is no value \( v \) such that for any input in the domain arbitrarily close to \((0, 0)\), the value of \( f \) gets arbitrarily close to \( v \); hence, \( f \) has no limit at \((0, 0)\).

Question 2 [40 points]
The regular octahedron, one of the five Platonic solids, is a 3-dimensional shape bounded by eight regular triangles such that four triangles meet in each vertex.

(a) Describe the symmetry group of the octahedron.
(b) Give a minimal set of generators for this group.

Assume the edges of the octahedron have length 1, and let \( d \) be the length of a long diagonal (the distance between two opposite vertices).

(c) Calculate \( d \).
(d) Is \( d \) constructible? If so, give a construction.
Let \( H \) be a plane through four of the vertices of the octahedron. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) describe how high above \( H \) the octahedron is.

(e) Give a formula for \( f \).
(f) Calculate the volume of the octahedron by integrating \( f \).
Solution

(a) The octahedron has 48 symmetries. These can be decomposed into 24 rotations (including the identity) and the 24 reflections of those symmetries. One way to describe them is as follows. The octahedron has 6 vertices; we can put any of these vertices at the top of the figure. Once the top is fixed, the bottom is also fixed, but the four remaining vertices still have some freedom: we can put any of these 4 vertices in the front. Once the front is fixed, the back is also fixed, but we can still choose which remaining vertex goes left and which goes right. This gives us $6 \times 4 \times 2 = 48$ elements of the group. The group structure is then given by the usual composition operator $\oplus$: given two symmetries, we obtain their composition by applying first the first transformation, and then the second.

(b) We need three elements to generate the group, for example, a rotation by 90° around the axis through the top and bottom vertices, a rotation by 90° around the axis through the front and back vertices, and any reflection. Any symmetry can be obtained by composing combinations of these three elements.

(c) To find the length $d$, we can observe that all the four edges forming a cycle in the horizontal plane (or any other axis-aligned plane) form a regular square. So, $d$ is the diagonal of a square of side length 1, which is $\sqrt{2}$.

(d) Yes, $\sqrt{2}$ is constructible.

(e) We align the vertices of the octahedron with the three axis of $\mathbb{R}^3$, and center it on the origin. Then we observe that $f$ is defined only for those values of $x$ and $y$ where $|x + y| \leq \frac{d}{2}$. We can express $f$ by $f(x, y) = \frac{d}{2} - |x| - |y|$, for $x, y$ in the domain.

(f) We express half of the volume of the octahedron by the integral

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \int_{|x| - \frac{d}{2}}^{\frac{d}{2} - |x|} \frac{d}{2} - |x| - |y| \, dy \, dx.$$  

To evaluate it, by symmetry, we observe we can restrict the domain to only positive values of $x$ and $y$. We find

$$\int_{0}^{\frac{d}{2}} \int_{0}^{\frac{d}{2} - x} \frac{d}{2} - x - y \, dy \, dx = \int_{0}^{\frac{d}{2}} \left[ \frac{d}{2} y - xy - \frac{1}{2} y^2 \right]_{0}^{\frac{d}{2} - x} \, dx = \int_{0}^{\frac{d}{2}} \frac{d^2}{8} - \frac{1}{2} dx + \frac{1}{2} x^2 \, dx$$

$$= \left[ \frac{1}{8} d^2 x - \frac{1}{4} dx^2 + \frac{1}{6} x^3 \right]_{0}^{\frac{d}{2}} = \frac{1}{48} d^3 = \frac{1}{24} \sqrt{2}.$$  

Since this is one quarter of the volume above the horizontal plane, it is one eighth of the total volume, which is, therefore, $\frac{1}{3} \sqrt{2}.$
**Question 3**

An embedding of a graph is called $k$-planar if each edge is intersected by at most $k$ other edges. A graph is $k$-planar if it has a $k$-planar embedding.

(a) Draw an example of a graph that is 1-planar, but not planar.
(b) What is the maximum number of edges in a 1-planar graph with $n$ vertices?

**Solution**

(a) For instance, the graph on the right. It is drawn 1-planar, but has too many edges to be planar: 8 vertices and 24 edges, which violates the bound $|E| \leq 3|V| - 6$ for planar graphs.

(b) First observe that in a maximal 1-planar graph, for any pair of crossing edges $ab$ and $cd$, we will also have the edges $ac$, $ad$, $bc$ and $bd$, and these edges do not cross any other edges. Indeed, if one of these edges is not present, we can simply add it in without creating any new crossings by closely following the edges $ab$ and $cd$.

Now, all crossing edges are bounded by cycles of four non-crossing edges, as is the case in the figure. Therefore, if we remove one of each crossing pair, we have a maximal (triangulated) planar graph, which has $3n - 6$ edges. Each removed edge spans two of these triangles, so the number of edges we removed is half the number of faces of the remaining triangulation. Using Euler’s formula, we see that a triangulation on $n$ vertices has $2n - 4$ faces, so we removed $n - 2$ edges. This means the total number of edges in the 1-planar graph was $4n - 8$. 