Exam Rules

1. The test consists of 7 questions on 4 pages, found on both sides of the paper.
2. You can get 100 points in total.
3. If you answer a question correctly, you will get the designated number of points. If you answer a question incorrectly, you will get no points. If you leave a (sub-)question unanswered, you will get 2 points.
4. Answer only in the designated boxes following the questions. All other sheets are scratch paper, and will not be checked. First solve the problems on scratch paper, and copy your solutions to this sheet.
5. No external material is allowed. You are provided with an auxiliary reference sheet with formulae.
6. Answer briefly, but completely; none of the questions in the exam require heavy computation or many details to reach a complete answer.
7. Good luck!

The Exam

1. (14 points) Using Euler-Poincaré formula, prove that pure quad meshes (all \( \text{deg}(f) = 4 \)) for closed (no boundary) meshes have average vertex valence of approximately 4 for small genus \( g \). What is the necessary genus of a mesh to be able to have exactly degree 4 on all vertices?

Tip: think about the total sum of vertex valences in a closed mesh.

Every face admits four edges, and every edge is represented in two faces. Therefore, we have \( 4F = 2E \). By the EP formula we have that \( V - E + F = \chi = 2 - 2g \), and then \( V = \chi + \frac{E}{2} \). Since the sum of valences is \( \sum_v \text{deg}(v) = 2E \) in a closed mesh (every edge contributes to two valences), we substitute \( \sum_v \text{deg}(v) = 4V - 2\chi \). Thus, since \( \chi = 2 - 2g \) is small, we get an average of \( \approx 4 \). The exact average is achieved where \( \chi = 0 = 2 - 2g \): for example, on a torus, where pure quad meshes are possible.
2. **(10 points)** Name one advantage and one disadvantage of cotangent-weight Laplacian (with the context of any application of your choice from the course). State an alternative formulation for a Laplacian which corrects the disadvantage you mentioned.

**Advantage:** This Laplacian is the correct geometric discrete Laplacian regarding the area gradient. Thus, it has linear precision (or: 2D reproducibility), and is less mesh dependent.

**Disadvantage:** the approximation breaks when the cotangent weights are negative. The Voronoi area becomes negative as well.

**Possible Solutions:** use other weights (like uniform Laplacian) to make it convex, or other Voronoi areas (like Barycentric or mixed cells) to make it positive definite again.

3. **(10 points)** Drawn is the control polyline for a cubic Bézier curve, consisting of points (2,1), (1,1), (2,0), and (3,1). Use de Casteljau to find the point on the curve at parameter value \( t = \frac{1}{4} \). Draw the construction and the resulting point, and write down its coordinates.

\[
(1\frac{19}{32}, \frac{55}{64})
\]

4. **(12 points)** Derive the blending functions for a quadratic B-spline using Cox-de Boor.

We know \( b_{i,1}(t) = 1 \) on \([i, i + 1]\), 0 otherwise.
We find \( b_{0,2}(t) = t \) on \([0, 1]\) and \( 2 - t \) on \([1, 2]\)
and \( b_{1,2}(t) = t - 1 \) on \([1, 2]\) and \( 3 - t \) on \([2, 3]\).
Finally, \( b_{0,3}(t) = \frac{1}{2}t^2 \) on \([0, 1]\), \( \frac{3}{2} - (t - \frac{3}{2})^2 \) on \([1, 2]\) and \( \frac{1}{2}(3 - t)^2 \) on \([2, 3]\).

Or, written out, \( b_{0,3}(t) = \frac{1}{2}t^2 \) on \([0, 1]\), \( -t^2 + 3t - \frac{3}{2} \) on \([1, 2]\) and \( \frac{1}{2}(3 - t)^2 \) on \([2, 3]\).

5. **(12 points)** What are the three main advantages/disadvantages of using axis-aligned versus freely oriented boxes as bounding volumes in a bounded volume hierarchy?

1: It is easier / cheaper to compute intersections between axis-aligned boxes than arbitrarily oriented boxes.
2: Arbitrarily oriented boxes will have smaller volume / less wasted space than axis-aligned boxes.
3: Axis-aligned boxes are only invariant under translations; arbitrarily oriented boxes are invariant under all rigid transformations.
6. (14 points) Consider the following sets $B$ of 4 black points at $(6,8)$, $(10,10)$, $(18,6)$ and $(24,4)$, and $W$ of 4 white points at $(8,12)$, $(12,14)$, $(16,16)$ and $(20,2)$.

(a) Give the Hausdorff distance and the Earth mover’s distance between $B$ and $W$. Draw the defining distances in the left copy of the figure.

Hausdorff distance: $H(B,W) = 6\sqrt{2}$
Earth mover’s distance: $E(B,W) = 6\sqrt{5} + 2\sqrt{26}$

(b) Draw a kD-tree on $B \cup W$ in the right copy of the figure.

7. (28 points) Answer 4 of these 8 questions concisely but with a full explanation. No formulas or calculation is needed. No bonus will be given on answering more than 4 questions, and only the first 4 answered will be graded. Simple yes-no answers will be disregarded even if they are correct.

(a) Is it possible to map a sphere to a cylinder without length or area distortions?

No, as they are not isometric-equivalent. This is evident since the cylinder has zero Gaussian curvature everywhere, and the sphere does not.

(b) How does the angle defect $G_v$ behave under surface scaling, as opposed to the continuous Gaussian curvature $K$, and why? how do you cancel this difference?

$G_v$ is an integrated quantity. That is, it is scale invariant, and only related to angles. Continuous Gaussian curvature is pointwise, which means it scales (quadratic inverse reciprocal). The difference is within the area element $dA$, so that in fact $KdA$ is approximated by $G_v$, and we use the Voronoi area $dA$ as a discrete area element.

(c) Explain how linear, constant precision, and positivity are important properties of barycentric coordinates for cage-based deformation?

Constant precision is the convex combination (partition of unity), which keeps the point inside a convex cage. Linear precision means that the deformed point $x'$ reproduces the original point $x$ when the cage $p$ reproduces the original cage $p'$, so the deformation is consistent. Positivity means that the deformation is intuitive, so that inner points move in the direction of the deformation of the cage and not in the other direction.
(d) What is the problem in Laplacian mesh deformation, that As-rigid-as-possible tries to solve?

Laplacian mesh deformation encodes details up to translation. That means that if the mesh is rotated, even globally, there is a shearing artefact. ARAP tries to reproduce the local 1-rings up to a rigid transformation, which is more general and detail preserving.

(e) How does the smoothness of reflection lines indicate the smoothness of the surface?

Reflection lines are continuous if and only if the normal is continuous, and that means $C^1$ continuity. If they are in addition smooth (continuous derivative), the surface is at least $C^2$ continuous.

(f) In a parameterization algorithm, what is the problem in the practice of cutting prior to parameterizing? what is a solution for it?

After cutting, nothing guarantees any continuity of scale or rotation across the cut, which creates an artefact. A solution to it is to flatten the mesh conceptually by moving curvature around, as in changing the angle defect (for instance), and then cutting. Methods like ABF or CETM do that.

(g) Why must the gradient of an energy $\nabla E$ be parallel to the gradient of a constraint $\nabla C$ in order to get a valid local critical point?

A critical point is where all the directional derivatives are zero in valid directions. Directional derivatives are zero in directions that are orthogonal to the gradient (unless it is zero on its own, and then all directional derivatives are zero). Thus, when the gradients of the energy and the constraints are parallel, valid directions, orthogonal to $\nabla C$, collide with zero directional derivatives of $E$.

(h) A student tried to solve the MLS equation in a point $p$, with a 2nd degree polynomial in $x, y, z$, and used 6 input surface (or off-surface) points. The system reported error. What is the problem, and how should it be fixed?

A 2nd degree polynomial admits more than 6 degrees of freedom. Therefore, the system was considerably underconstrained, with many solutions. Algebraically, the matrix was only semidefinite. The solution is to use more than the minimal amount of points, which is $|\{1, x, y, z, xy, yz, xz, x^2, y^2, z^2\}| = 10$. 