Multi-attribute utility theory
(very superficial in textbook!)

Multiple objectives – an example
Consider the problem of deciding upon treatment for a patient with esophageal cancer. The problem can be (schematically) modelled as follows:

When a decision problem concerns multiple objectives, captured by multiple attributes, consequences are no longer 'simple'; this is apparent from the above consequence matrix.

Multiple objectives – another example
The City of Utrecht is considering four different sites (A, B, C, D) for a new electric power generating station.

The objectives of the city are to
• minimise the cost of building the station;
• minimise the acres of land damaged by building it.

Factors influencing the objectives include the land type at the different sites, the architect and construction company hired, the cost of material and machines used, the weather, etc.

Costs, however, are estimated to fall between €15 million and €60 million; between 200 and 600 acres of land will be damaged.

The possible consequences of the decision alternatives are captured by two attributes. We therefore need to determine a two-attribute utility function: \( u(\text{Cost, Acres}) \) (or \( u(C, A) \)).

The multiattribute utility function – assessment
Let \( X_1, \ldots, X_n, n \geq 2 \), be a set of attributes associated with the consequences of a decision problem.

The utility of a consequence \((x_1, \ldots, x_n)\) can be determined from

1. **direct assessment:** estimate the combined utility \( u(x_1, \ldots, x_n) \) over the given values of all \( n \) attributes;

2. **decomposed assessment:**
   1. estimate \( n \) conditional utilities \( u_i(x_i) \) for the given values of the \( n \) attributes;
   2. compute \( u(x_1, \ldots, x_n) \) by combining the \( u_i(x_i) \) of all attributes:

\[
u(x_1, \ldots, x_n) = f[u_1(x_1), \ldots, u_n(x_n)]\]
Assign a utility of 0 to the worst consequence (60, 600), and a utility of 1 to the best consequence (15, 200).

The utility of, for example, consequence (50, 300) can be determined from:

\[
\begin{align*}
0 & \sim (50,300) \\
1 & \sim (15,200) \\
0 & \sim (60,600)
\end{align*}
\]

To find a good representation through direct assessment, utilities must be assessed for a substantial number of points.

Different functional forms

Let \( X \) and \( Y \) be two attributes (generalisation to \( n > 2 \) attributes is straightforward). Consider the utility function \( u(X,Y) \) for \( X \) and \( Y \).

- \( u \) has an additive form if for constants \( k_X \) and \( k_Y \):
  \[
  u(X,Y) = k_X \cdot u_X(X) + k_Y \cdot u_Y(Y)
  \]
- \( u \) has a multilinear form if for constants \( k_X, k_Y, k_{XY} \):
  \[
  u(X,Y) = k_X \cdot u_X(X) + k_Y \cdot u_Y(Y) + k_{XY} \cdot u_X(X) \cdot u_Y(Y)
  \]
- \( u \) has a multiplicative form if for constants \( k_X, k_Y, c_X, \) and \( c_Y \):
  \[
  u(X,Y) = (k_X \cdot u_X(X) + c_X) \cdot (k_Y \cdot u_Y(Y) + c_Y)
  \]

The constants are often called weights or scaling constants.

The different functional forms are only valid under certain assumptions; the values of the scaling constants are thereby often constrained!

Notation

Let \( X \) be an attribute and \( Y \) a set of \( n - 1, n > 0 \), attributes.

Compare the following:

- if \( n = 1 \) then \( u(X) \) is a utility function for \( X \) in a one-dimensional decision problem
- if \( n > 1 \) then
  - \( u(X,Y) \) is an \( n \)-attribute utility function;
  - \( u(X,y_i) \) is a subutility function for \( X \) given a fixed value-assignment \( y_i \) to attributes \( Y \);
  - \( u_X(X) \) is a conditional utility function for \( X \): a (possibly) re-scaled function \( u(X,y_k) \) for some — no longer explicit — \( y_k \).

An example

The City of Utrecht decides to model the utility function \( u(Cost, Acres) \) as an additive function.

Using standard assessment techniques, they find for the conditional utility functions \( u_C \) and \( u_A \) that:

\[
\begin{align*}
  u_C(15) = 1.0 & \quad u_C(30) = 0.5 & \quad u_C(50) = 0.2 & \quad u_C(60) = 0.0 \\
  u_A(200) = 1.0 & \quad u_A(300) = 0.8 & \quad u_A(400) = 0.5 & \quad u_A(600) = 0.0
\end{align*}
\]

The City finds Cost three times as important as Acres lost. The resulting utilities for a number of Cost-Acres pairs are then:

\[
\begin{align*}
  u(50,300) &= 3 \cdot u_C(50) + 1 \cdot u_A(300) = 3 \cdot 0.2 + 1 \cdot 0.8 = 1.4 \\
  u(30,400) &= 1.5 + 0.5 = 2.0 \\
  u(60,200) &= 0.0 + 1.0 = 1.0 \\
  u(15,600) &= 3.0 + 0.0 = 3.0 \\
  u(15,200) &= 3.0 + 1.0 = 4.0
\end{align*}
\]
Interpreting weights or scaling constants
Reconsider the City of Utrecht’s two attribute utility function
\[ u(C, A) = 3 \cdot u_C(C) + 1 \cdot u_A(A). \]
Suppose that the City has explicitly expressed the indifference \((50, 600) \sim (60, 350)\), which indeed holds given all current functions:
\[
\begin{align*}
    u(50, 600) &= 3 \cdot u_C(50) + u_A(600) = 3 \cdot 0.2 + 0 = 0.6 \\
    u(60, 350) &= 3 \cdot u_C(60) + u_A(350) = 3 \cdot 0 + 0.6 = 0.6
\end{align*}
\]
Now, however, the City finds out that all alternatives result in at least a loss of 350 acres of land, and we rescale \(u_C\) such that \(u_C(350) = 1\). The expressed indifference still holds, implying that
\[ u(60, 350) = k_C \cdot 0 + k_A \cdot 1 = u(50, 600) = k_C \cdot 0.2 + k_A \cdot 0 \]
From this it follows that \(k_C = 5 \cdot k_A\).
Did \(C\) just turn from three times as important as \(A\) to five times as important?

Pricing out – an example
Suppose the following utilities are assessed by the City of Utrecht for Cost and Acres lost:
\[
\begin{align*}
    u_C(15) &= 1.0 & u_C(30) &= 0.7 & u_C(50) &= 0.4 & u_C(60) &= 0.0 \\
    u_A(200) &= 1.0 & u_A(300) &= 0.8 & u_A(400) &= 0.5 & u_A(600) &= 0.0
\end{align*}
\]
The City decides that it is willing to sacrifice 200 acres of land if that would save \(€10\) million. This implies that, for example,
\[ u(40, 200) = u(30, 400) \]
that is, \(a \cdot u_C(40) + b \cdot u_A(200) = a \cdot u_C(30) + b \cdot u_A(400)\)
We conclude that
\[
\begin{align*}
    a &= \frac{u_A(400) - u_A(200)}{u_C(40) - u_C(30)} = \frac{0.5 - 1.0}{0.4 - 0.7} = -0.5 \\
    b &= \frac{u_C(400) - u_C(30)}{u_C(40) - u_C(30)} = \frac{0.5}{0.4 - 0.7} = -0.3
\end{align*}
\]
And thus find that
\[ u(C, A) = 0.5 \cdot u_C(C) + 0.3 \cdot u_A(A) \]
Is this valid?

Assessing scaling constants (I)

Pricing out:
Assess the marginal rate of substitution, that is, determine the value of one objective in terms of another.
Let \(X\) and \(Y\) be two attributes with values \(x_1 \leq \ldots \leq x_n, n \geq 2\), and \(y_1 \leq \ldots \leq y_m, m \geq 2\). Let \(e_X\) and \(e_Y\) be the units of measurement for the two attributes, respectively.
Suppose you are willing to sacrifice \(s\) units of \(Y\) for 1 unit of \(X\). Then for all \(x_i, y_j\):
\[ u(x_i, y_j) = u(x_i + e_X, y_j - s \cdot e_Y) \]
If \(u(X,Y)\) is additive and \(u_X(X), u_Y(Y)\) are linear functions, then this implies that
\[ a \cdot u_X(x_i) + b \cdot u_Y(y_j) = a \cdot u_X(x_i + e_X) + b \cdot u_Y(y_j - s \cdot e_Y) \]
As a result,
\[
\begin{align*}
    a &= \frac{u_Y(y_j - s \cdot e_Y) - u_Y(y_j)}{u_X(x_i + e_X) - u_X(x_i)} \\
    b &= \frac{u_Y(y_j) - u_Y(y_j - s \cdot e_Y)}{u_X(x_i + e_X) - u_X(x_i)}
\end{align*}
\]

Assessing scaling constants (II)

Swing weighting:
Consider a set of \(n \geq 2\) attributes \(X_1, \ldots, X_n\). Swing weighting assigns weights to attributes based on either a rank-order or a rating on attributes and one consequence.
Take the (theoretically) worst possible consequence as benchmark. Then apply the following procedure:
rate(benchmark) ← 0 ;
rank(benchmark) ← \(n + 1\) ;
for \(i = 1\) to \(n\) do
    \(Z \leftarrow \text{answer to: if you could swing one attribute from worst to best value, which would you swing?} \)
    Swing(Z);
    rank(Z) ← \(i\).
    if \(i = 1\) then rate(Z) ← 100
    else rate(Z) ← answer ∈ (0, 100).
Swing weighting – cntd

Consider a set of \( n \geq 2 \) attributes \( X_1, \ldots, X_n \). Let \( \text{rank}(X_i) \) and \( \text{rate}(X_i) \) denote a ranking and a rating for attribute \( X_i \), respectively.

The weight \( w(X_i) \) for attribute \( X_i \) can now be determined using either of the following two approaches:

- direct rating: \( w(X_i) = \frac{\text{rate}(X_i)}{\sum_{j=1}^{n} \text{rate}(X_j)} \)
- rank-sum weighing:
  \[
  w(X_i) = \frac{n - \text{rank}(X_i) + 1}{n + 1}
  \]

Swing weighting – an example

<table>
<thead>
<tr>
<th>Attribute to swing</th>
<th>consequence</th>
<th>rank</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>(benchmark)</td>
<td>60 600</td>
<td>2+1</td>
<td>0</td>
</tr>
<tr>
<td>Cost</td>
<td>15 600</td>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>Acres</td>
<td>60 200</td>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>total:</td>
<td></td>
<td></td>
<td>130</td>
</tr>
</tbody>
</table>

**direct rating:**
\[
\begin{align*}
  w(C) &= \frac{100}{130} = 0.77 \\
  w(A) &= \frac{30}{130} = 0.23
\end{align*}
\]

**rank-sum weighting:**
\[
\begin{align*}
  w(C) &= \frac{2 - 1 + 1}{3} = \frac{2}{3} \\
  w(A) &= \frac{2 - 2 + 1}{3} = \frac{1}{3}
\end{align*}
\]

Assessing scaling constants (III)

**Lottery weights:**

Consider two attributes \( X \) and \( Y \) (the following extends straightforwardly to \( n > 2 \) attributes).

Let \( (x^0, y^0) \) denote the worst possible consequence, and \( (x^+, y^+) \) the best possible consequence.

The weight \( w(X) \) for attribute \( X \) equals \( p \), where \( p \) is the indifference probability that follows from:

\[
1.0 \ (x^+, y^+) \sim p \ (x^0, y^0) \sim (1-p) \ (x^+, y^+)
\]

Lottery weights: an example

Assume once more that the City of Utrecht decides to model the utility function \( u(\text{Cost}, \text{Acres}) \) as an additive function:

\[
u(C, A) = k_C \cdot u_C(C) + k_A \cdot u_A(A)
\]

The weight \( k_C \) is assessed from:

\[
\begin{align*}
  1.0 \ (15, 200) &\sim (1-k_C) \ (60, 600)
\end{align*}
\]

The weight \( k_A \) is assessed using:

\[
\begin{align*}
  1.0 \ (60, 200) &\sim (1-k_A) \ (60, 600)
\end{align*}
\]
MAUT with \( n = 2 \) attributes: when can we use additive and multilinear forms?

Additive independence – the formal definition

Use of an additive utility function is justified given the assumption of additive independence [AI].

Two attributes \( X \) and \( Y \) are additive independent if preferences for lotteries over \( X \times Y \) can be established by comparing the values one attribute at a time. More formally,

**Definition**

Two attributes \( X \) and \( Y \) are additive independent if the paired preference comparison of any two lotteries, defined by two joint probability distributions on \( X \times Y \), depends only on their marginal distributions.

Interpreting additive independence

The ability to establish preferences for lotteries over \( X \times Y \) by comparing the values one attribute at a time entails the following:

- the decisionmaker should be indifferent between \([p, (x_1, y_1); (1 - p), (x_2, y_2)]\) and \([p, (x_1, y_2); (1 - p), (x_2, y_1)]\) since both have the same probability of achieving \( x_1 \) vs \( x_2 \);
- the decisionmaker should also be indifferent between \([p, (x_1, y_1); (1 - p), (x_2, y_2)]\) and \([p, (x_2, y_1); (1 - p), (x_1, y_2)]\) since both have the same probability of achieving \( y_1 \) vs \( y_2 \);
- this can only hold if \( p = 1 - p = 0.5 \)

Additive independence – a practical definition

**Definition**

Consider two attributes \( X \) and \( Y \) with values \( x_1, \ldots, x_n, n \geq 2 \), and \( y_1, \ldots, y_m, m \geq 2 \).

\( X \) and \( Y \) are additive independent if for an arbitrary pair \((x_j, y_l)\), we have for all pairs \((x_i, y_k)\) that

\[
\begin{array}{ccc}
0.5 & (x_j, y_l) & 0.5 \\
0.5 & (x_i, y_k) & \sim \\
0.5 & (x_j, y_l) & 0.5 \\
0.5 & (x_i, y_k) & \\
\end{array}
\]

Note that additive independence is a symmetric property.
Additive independence – an example

Let \( u(X, Y) \) be a two-attribute utility function defined as follows:

\[
\begin{align*}
 u(x_0, y_0) &= 1.0 & u(x_0, y_1) &= 0.7 & u(x_0, y_2) &= 0.5 \\
 u(x_1, y_0) &= 0.9 & u(x_1, y_1) &= 0.6 & u(x_1, y_2) &= 0.4 \\
 u(x_2, y_0) &= 0.5 & u(x_2, y_1) &= 0.2 & u(x_2, y_2) &= 0.0
\end{align*}
\]

\( X \) and \( Y \) are additive independent:

\[
\begin{align*}
 [0.5, (x_1, y_1); 0.5, (x_0, y_0)] &\sim [0.5, (x_1, y_0); 0.5, (x_0, y_1)] \\
 [0.5, (x_2, y_1); 0.5, (x_0, y_0)] &\sim [0.5, (x_2, y_0); 0.5, (x_0, y_1)] \\
 [0.5, (x_1, y_2); 0.5, (x_0, y_0)] &\sim [0.5, (x_1, y_0); 0.5, (x_0, y_2)] \\
 [0.5, (x_2, y_2); 0.5, (x_0, y_0)] &\sim [0.5, (x_2, y_0); 0.5, (x_0, y_2)] \\
 \ldots
\end{align*}
\]

Additive independence – the implication for \( u(X, Y) \)

Consider two attributes \( X \) and \( Y \) with values \( x_1, \ldots, x_n, n \geq 2, \) and \( y_1, \ldots, y_m, m \geq 2. \)

If \( X \) and \( Y \) are additive independent, then for any pair \( (x_j, y_l), \) and all pairs \( (x_i, y_k), \) we have by definition that

\[
0.5 \cdot u(x_j, y_k) + 0.5 \cdot u(x_j, y_l) = 0.5 \cdot u(x_j, y_l) + 0.5 \cdot u(x_j, y_k)
\]

that is, the change in utility for values of one attribute is independent of the values of the other attribute:

\[
u(x_i, y_k) - u(x_j, y_k) = u(x_i, y_l) - u(x_j, y_l)
\]

This means that all subutility functions \( u(X, y_k) \) are the same, up to translation: \( u(X, y_k) = u(X, y_l) + c_{kl} \) for some constant \( c_{kl}. \)

Exploiting additive independence – an example

To establish \( u(C, A) \), the City of Utrecht assesses

- \( u(C, 600) \) for fixed acres:
  - \( u(15, 600) = 0.75 \)
  - \( u(50, 600) = 0.15 \)
  - \( u(30, 600) = 0.35 \)
  - \( u(60, 600) = 0.00 \)

- and \( u(60, A) \) for fixed cost:
  - \( u(60, 200) = 0.25 \)
  - \( u(60, 400) = 0.15 \)
  - \( u(60, 300) = 0.20 \)
  - \( u(60, 600) = 0.00 \)

These two functions intersect at \( u(60, 600) = 0. \)

Assuming additive independence, we have that for all \( c_i \) and \( a_j:\)

\[
u(c_i, a_j) + u(60, 600) = u(c_i, 600) + u(60, a_j)
\]

All other points can now be computed. For example,

\[
u(15, 200) = u(15, 600) + u(60, 200) - 0 = 1.0
\]
The additive utility function – Ia

**Theorem**
Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n$, $n \geq 2$, and $y_1 \leq \ldots \leq y_m$, $m \geq 2$. Attributes $X$ and $Y$ are additive independent if the two-attribute utility function is additive:

$$u(X, Y) = k_X \cdot u_X(X) + k_Y \cdot u_Y(Y),$$

where

- $u(X, Y)$ is normalised with $u(x_1, y_1) = 0$ and $u(x_n, y_m) = 1$;
- $u_X(X)$ is a conditional utility function on $X$, normalised by $u_X(x_1) = 0$ and $u_X(x_n) = 1$;
- $u_Y(Y)$ is a conditional utility function on $Y$, normalised by $u_Y(y_1) = 0$ and $u_Y(y_m) = 1$;
- $k_X = u(x_n, y_1)$ and $k_Y = u(x_1, y_m)$ are positive scaling constants, summing to 1.

Analogous to proof of lemma.

**Proof of Lemma:**

$(\Rightarrow)$ Let $x_i$ and $y_k$ be arbitrary values of $X$ resp. $Y$.

Additive independence implies

$$u(x_1, y_1) + u(x_i, y_k) = u(x_1, y_k) + u(x_i, y_1)$$

Setting $u(x_1, y_1) = 0$, we get $u(x_i, y_k) = u(x_1, y_k) + u(x_i, y_k)$.

$(\Leftarrow)$ Suppose $u(X, Y) = u(X, y_1) + u(x_1, Y)$, and let $x_j$ and $y_l$ be arbitrary values of $X$ resp. $Y$.

Then for all values $x_i$ and $y_k$ of $X$ resp. $Y$, we have that

$$u(x_i, y_k) + u(x_j, y_l) = u(x_i, y_k) + u(x_i, y_k) + u(x_j, y_l) + u(x_1, y_l)$$

$$= u(x_i, y_k) + u(x_1, y_l) + u(x_j, y_l) + u(x_1, y_l)$$

$$= u(x_i, y_k) + u(x_j, y_l).$$

We conclude that $X$ and $Y$ are additive independent. ■

The additive utility function – IIa

**Lemma**
Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n$, $n \geq 2$, and $y_1 \leq \ldots \leq y_m$, $m \geq 2$.

Attributes $X$ and $Y$ are additive independent iff the two-attribute utility function is additive:

$$u(X, Y) = u(X, y_1) + u(x_1, Y),$$

where $u(x_1, y_1) = 0$.

The additive utility function – IIb

**Proof of Lemma:**

$(\Rightarrow)$ From the previous lemma we have that for arbitrary values $x_i$ and $y_k$ of $X$ resp. $Y$

$$u(x_i, y_k) = u(x_i, y_k) + u(x_i, y_k) \text{ if } u(x_1, y_1) = 0$$

Normalising $u(x_1, Y)$ gives us $u(x_1, Y) = k_Y \cdot u_Y(Y)$ with $k_Y = u(x_1, y_m)$;

similarly, $u(X, y_1) = k_X \cdot u_X(X)$ with $k_X = u(x_n, y_1)$.

$(\Leftarrow)$ Analogous to proof of lemma. ■

The additive utility function – Iib

**Proof of Theorem:**

$(\Rightarrow)$ Let $x_i$ and $y_k$ be arbitrary values of $X$ resp. $Y$.

Additive independence implies

$$u(x_i, y_k) = u(x_i, y_k) + u(x_i, y_k)$$

Normalising $u(x_i, Y)$ gives us $u(x_i, Y) = k_Y \cdot u_Y(Y)$ with $k_Y = u(x_1, y_m)$;

similarly, $u(X, y_1) = k_X \cdot u_X(X)$ with $k_X = u(x_n, y_1)$.

$(\Leftarrow)$ Analogous to proof of lemma. ■
Assessing the additive utility function – an example

Suppose the City of Utrecht assesses the following conditional utilities for Cost and Acres lost:

\[ u_C(15) = 1.0 \quad u_A(200) = 1.0 \]
\[ u_C(30) = 0.5 \quad u_A(300) = 0.8 \]
\[ u_C(50) = 0.2 \quad u_A(400) = 0.5 \]
\[ u_C(60) = 0.0 \quad u_A(600) = 0.0 \]

For scaling constants \( k_A \) and \( k_C \) we know that \( k_A = 1 - k_C \) and that \( k_C = u(15, 600) \). This latter utility is assessed using the following lottery:

\[
1.0 \quad \begin{array}{c}
\text{\(15, 600\)} \\
\sim \\
\text{\((1 - k_C) \quad \text{\(60, 600\)}\)}
\end{array}
\]

The indifference probability \( k_C \) is found to be 0.75. We therefore conclude that

\[ u(C, A) = 0.75 \cdot u_C(C) + 0.25 \cdot u_A(A) \]

Interpreting utility independence

Independence of conditional preferences for lotteries over \( X \) of the value of \( Y \), entails the following:

- if \([p, (x_1, y_1)]; (1 - p), (x_2, y_1) \] \( \succeq [p, (x_3, y_1)]; (1 - p), (x_4, y_1) \]
  then the decision maker should also prefer \([p, (x_1, y_2)]; (1 - p), (x_2, y_2) \] over \([p, (x_3, y_2)]; (1 - p), (x_4, y_2) \)
  since there is only a change in sure outcome of attribute \( Y \)
  \( (y_1 \text{ vs } y_2) \), which should not affect preferences among
  lotteries over \( X \);
- similarly, if \((x_c, y_1)\) is the certainty equivalent of the first
  lottery, then \((x_c, y_2)\) should be the certainty equivalent of the
  third lottery;

Note that the above means that also the preference order on values of \( X \) should be independent of the value of \( Y \).

Utility independence – the formal definition

Use of a multiplicative or additive utility function is justified under (mutual) utility independence [(M)UI].

Attribute \( X \) is utility independent of attribute \( Y \) if conditional preferences for lotteries over \( X \) given a fixed value for \( Y \) do not depend on the particular value for \( Y \). More formally,

\[ \text{An attribute } X \text{ is utility independent of an attribute } Y \text{ iff for any} \]

\[ \text{lotteries } [(X, y_k)_1] \text{ and } [(X, y_k)_2] \text{ over } X \times Y \text{ with } Y \text{ fixed to value } y_k, \]

\[ \text{we have} \]

\[ [(X, y_k)_1 \succeq [(X, y_k)_2] \implies [(X, y)_1 \succeq [(X, y)_2] \forall y \text{ of } Y \]

NB: \([(X, y)] \) represents a conditional lottery over \( X \times Y \) involving consequences over different values of \( X \) combined with a fixed value for \( Y \).

Utility independence – a practical definition

Consider two attributes \( X \) and \( Y \) with values \( x_1, \ldots, x_n, n \geq 2, \) and \( y_1, \ldots, y_m, m \geq 2. \)

\( X \) is utility independent of \( Y \) iff for an arbitrary triple \( x_i, x_j, x_k \) with \( x_i \preceq x_j \preceq x_k \) there exists a probability \( p \) such that for all

\[ \text{values } y_i \text{ of } Y, \text{ we have that} \]

\[
1.0 \quad \begin{array}{c}
\text{\((x_j, y_i)\)} \\
\sim \\
\text{\((1 - p) \quad \text{\((x_k, y_i)\)}\)}
\end{array}
\]

If \( X \) is utility independent of \( Y \) and \( Y \) is utility independent of \( X \)

\[ \text{then } X \text{ and } Y \text{ are mutually utility independent.} \]

Note: the above definition, in terms of a probability equivalent, can also be rephrased in terms of a certainty equivalent for a 50-50 lottery.
An example

Let \( u(X, Y) \) be a two-attribute utility function defined as follows:

\[
\begin{align*}
  u(x_0, y_0) &= 1.0, \\
  u(x_1, y_0) &= 0.6, \\
  u(x_2, y_0) &= 0.0.
\end{align*}
\]

Then we have:

- \( X \) is utility independent of \( Y \):
  - \([1.0, (x_1, y_0)] \sim [0.6, (x_0, y_0); 0.4, (x_2, y_0)]\)
  - \([1.0, (x_1, y_1)] \sim [0.6, (x_0, y_1); 0.4, (x_2, y_1)]\)
  - \([1.0, (x_1, y_2)] \sim [0.6, (x_0, y_2); 0.4, (x_2, y_2)]\)

- \( Y \) is not utility independent of \( X \), as in the context of \( x_1 \) we have \( y_1 \succ y_0 \succ y_2 \), and in the context of \( x_2 \) we have \( y_1 \succ y_2 \succ y_0 \)!

Utility independence – the implication for \( u(X, Y) \)

Consider two attributes \( X \) and \( Y \) with values \( x_1, \ldots, x_n, n \geq 2 \), and \( y_1, \ldots, y_m, m \geq 2 \).

First we observe that \( x_1 \leq x_j \leq x_n \) for each value \( x_j \) of \( X \).

Now, if \( X \) is utility independent of \( Y \), then we have by definition that for any \( x_j \) there exists a \( p_j \) such that for all \( y_l \),

\[
u(x_j, y_l) = p_j \cdot u(x_1, y_l) + (1 - p_j) \cdot u(x_n, y_l)
\]

where

\[
p_j = \frac{u(x_j, y_l) - u(x_m, y_l)}{u(x_1, y_l) - u(x_m, y_l)}
\]

This means that all subutility functions \( u(X, y_l) \) are the same, up to (positive) scaling and translation:

\[
u(X, y_l) = c_{kl} \cdot u(X, y_l) + d_{kl} \text{ for some constants } c_{kl} > 0 \text{ and } d_{kl}.
\]

Proof of the Proposition (sketch):

1. \((\Rightarrow)\) First observe that for all \( x, x_1 \leq x \leq x_n \). Utility independence holds iff for each \( x \) a probability \( p \) exists such that

\[
u(X, Y) \sim [p, (x_1, Y)]; (1 - p), (x_n, Y)\]

and therefore (main theorem)

\[
u(X, Y) = p \cdot u(x_1, Y) + (1 - p) \cdot u(x_n, Y)
\]

1. solve \( p \) from (I) with \( Y \) set to \( g_i \);
2. substitute this result in (I) to get the desired result.

1. \((\Leftarrow)\) Let \( g_i > 0 \) and \( h_i \) be such that for arbitrary \( y_i \):

\[
u(X, Y) = g_i(Y) \cdot u(X, y_i) + h_i(Y)
\]

1. for arbitrary \( x_j \) and \( y_j \), choose \( x_1 \) and \( x_k \) s.t.

\[
(x_i, y_j) \preceq (x_j, y_j) \preceq (x_k, y_j);
\]
2. continuity: \( \exists \ p: u(x_j, y_j) = p \cdot u(x_i, y_j) + (1 - p) \cdot u(x_k, y_j); \)
3. apply (II) to \( u(x_1, y_j), u(x_j, y_j) \) and \( u(x_k, y_j) \).
4. rewrite to find the desired result.
Exploiting utility independence

Consider two attributes $X$ and $Y$ with values $x_1, \ldots, x_n$, $n \geq 2$, and $y_1, \ldots, y_m$, $m \geq 2$.

If $X$ is utility independent of $Y$, then one subutility function $u(X, y_k)$ and two points $u(x_i, y_l)$ and $u(x_j, y_l)$ serve for establishing a second function: $u(X, y_l)$

$u(X, Y)$ can be constructed entirely from three subutility functions:

1. $u(X, y_k)$ for some value $y_k$ of $Y$
2. $u(x_i, Y)$ for some value $x_i$ of $X$
3. $u(x_j, Y)$ for some value $x_j$ of $X$, $x_j \neq x_i$

Three subutility functions

**Theorem**

Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n$, $n \geq 2$, and $y_1 \leq \ldots \leq y_m$, $m \geq 2$.

If $X$ is utility independent of $Y$ then

$$u(X, Y) = u(x_1, Y) \cdot [1 - u(X, y_1)] + u(x_n, Y) \cdot u(X, y_1)$$

where $u(X, Y)$ is normalised by $u(x_1, y_1) = 0$ and $u(x_n, y_1) = 1$.

An Example

Suppose the City of Utrecht assesses the following utilities for Cost and Acres lost, given $A$ fixed at 600, and $C$ fixed at 15 and 60, respectively:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$C$</th>
<th>$u(15, 600)$</th>
<th>$u(15, 200)$</th>
<th>$u(60, 200)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>60</td>
<td>0.75</td>
<td>1.00</td>
<td>0.20</td>
</tr>
<tr>
<td>30</td>
<td>60</td>
<td>0.50</td>
<td>0.90</td>
<td>0.15</td>
</tr>
<tr>
<td>50</td>
<td>60</td>
<td>0.10</td>
<td>0.80</td>
<td>0.10</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>0.00</td>
<td>0.75</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Cost is utility independent of Acres lost. We normalise $u(C, A)$ such that $u(60, 600) = 0$ and $u(15, 600) = 1$:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$C$</th>
<th>$u(15, 600)$</th>
<th>$u(15, 200)$</th>
<th>$u(60, 200)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>60</td>
<td>1.00</td>
<td>1.33</td>
<td>0.27</td>
</tr>
<tr>
<td>30</td>
<td>60</td>
<td>0.67</td>
<td>1.20</td>
<td>0.20</td>
</tr>
<tr>
<td>50</td>
<td>60</td>
<td>0.13</td>
<td>1.07</td>
<td>0.13</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>0.00</td>
<td>1.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Now,

$$u(C, A) = u(60, A) \cdot [1 - u(C, 600)] + u(15, A) \cdot u(C, 600)$$
**The additive utility function — IIIa**

**Corollary**
Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n, n \geq 2,$ and $y_1 \leq \ldots \leq y_m, m \geq 2.$

If $X$ is utility independent of $Y,$ then the two-attribute utility function is additive iff
\[
[0.5, (x_1, y_1); 0.5, (x_n, Y)] \sim [0.5, (x_1, Y); 0.5, (x_n, y_1)]
\]

Under the above conditions, we have that
\[
u(X, Y) = \nu(x_1, Y) + \nu(X, y_1)\]
where $\nu(x_1, y_1) = 0$ and $\nu(x_n, y_1) = 1.$

---

**Proof of the Corollary:**
Utility independence implies that
\[
u(X, Y) = \nu(x_1, Y) [1 - \nu(x_n, Y) + \nu(X, y_1) \cdot \nu(X, y_1)]
\]
with $\nu(x_1, y_1) = 0$ and $\nu(x_n, y_1) = 1.$

The lottery equivalence translates into
\[
0 + \nu(x_n, Y) = 1 + \nu(x_1, Y)
\]
Substitute $\nu(x_n, Y)$ with $1 + \nu(x_1, Y)$ in the above results in
\[
\nu(X, Y) = \nu(x_1, Y) + \nu(X, y_1)
\]
with $\nu(x_1, y_1) = 0$ and $\nu(x_n, y_1) = 1.$

■

---

**The multilinear utility function — Ia**

**Corollary**
Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n, n \geq 2,$ and $y_1 \leq \ldots \leq y_m, m \geq 2.$

If $X$ is utility independent of $Y,$ then the two-attribute utility function is multilinear iff
\[
u(x_n, Y) = 1 + b \cdot \nu(x_1, Y)
\]
for some constant $b > 0.$

Under the above conditions, we have that
\[
u(X, Y) = \nu(x_1, Y) + \nu(X, y_1) + (b - 1) \cdot \nu(x_1, Y) \cdot \nu(X, y_1)
\]
where $\nu(x_1, y_1) = 0$ and $\nu(x_n, y_1) = 1.$
Exploiting mutual utility independence

Consider two attributes $X$ and $Y$ with values $x_1, \ldots, x_n$, $n \geq 2$, and $y_1, \ldots, y_m$, $m \geq 2$.

If $X$ and $Y$ are mutually utility independent, then $u(X,Y)$ can be constructed from two subutility functions and one point:
1. $u(X, y_k)$ for some value $y_k$ of $Y$
2. $u(x_i, Y)$ for some value $x_i$ of $X$
3. $u(x_j, y_l)$ for some values $x_j \neq x_i$ of $X$, and $y_l \neq y_k$ of $Y$

if $X$ is utility independent of $Y$, but $Y$ is utility dependent

if $X$ and $Y$ are mutually utility independent:

\[ u(x, y) = g(x) \cdot h(y) \]

for some utility functions $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$.

In addition, the utility $u(x, y)$ is UI of $X$, $Y$, $u(x, y)$ is UI of $X, Y$. Mutual utility independence implies strategical equivalence of all 'horizontal' functions with $u(C, 600)$ and all 'vertical' functions with $u(60, A)$:

- we first determine the function $u(C, 300)$;
- we then use $u(C, 300)$ to derive all 'vertical' functions from $u(60, A)$.

Example – continued (II)

The utilities assessed by the City of Utrecht:

\[
\begin{align*}
 u(15, 600) &= 0.75 & u(30, 600) &= 0.35 & u(50, 600) &= 0.15 & u(60, 600) &= 0.15 \ 
 u(60, 200) &= 0.25 & u(60, 300) &= 0.20 & u(60, 400) &= 0.15 & u(60, 400) &= 0.00
\end{align*}
\]

In addition, the utility $u(30, 300) = 0.30$ is assessed.

- we first determine the function $u(C, 300)$: since $C$ is UI of $A$, there exist functions $g$ and $h$ such that e.g.:

\[
u(C, 300) = g_{600}(300) \cdot u(C, 600) + h_{600}(300)\]

We require two equations to find $g_{600}(300)$ and $h_{600}(300)$:

\[
\begin{align*}
u(30, 300) &= 0.30 = g_{600}(300) \cdot u(30, 600) + h_{600}(300) \\
&= g_{600}(300) \cdot 0.35 + h_{600}(300) \\
u(60, 300) &= 0.20 = g_{600}(300) \cdot 0.00 + h_{600}(300) \\
\text{resulting in } u(C, 300) &= 0.29 \cdot u(C, 600) + 0.20
\end{align*}
\]

We require two equations to find $f_{60}(C)$ and $k_{60}(C)$:

\[
\begin{align*}
u(C, 300) &= f_{60}(C) \cdot u(60, 300) + k_{60}(C) \\
u(C, 600) &= f_{60}(C) \cdot u(60, 600) + k_{60}(C)
\end{align*}
\]

Given that $u(C, 300)$ and $u(C, 600)$ are known functions, we can compute an $f_{60}(c_i)$ and $k_{60}(c_i)$ for each $c_i$. 

The utilities assessed by the City of Utrecht:

\[
\begin{align*}
u(15, 600) &= 0.75 & u(30, 600) &= 0.35 & u(50, 600) &= 0.15 & u(60, 600) &= 0.15 \ 
 u(60, 200) &= 0.25 & u(60, 300) &= 0.20 & u(60, 400) &= 0.15 & u(60, 400) &= 0.00
\end{align*}
\]

In addition, the utility $u(30, 300) = 0.30$ is assessed.

- we first determine the function $u(C, 300)$;
- we then use $u(C, 300)$ to derive all 'vertical' functions from $u(60, A)$.

\[
\begin{align*}
u(15, 600) &= 0.75 & u(30, 600) &= 0.35 & u(50, 600) &= 0.15 & u(60, 600) &= 0.15 \ 
 u(60, 200) &= 0.25 & u(60, 300) &= 0.20 & u(60, 400) &= 0.15 & u(60, 400) &= 0.00
\end{align*}
\]

In addition, the utility $u(30, 300) = 0.30$ is assessed.

- we then use $u(C, 300)$ to derive all 'vertical' functions from $u(60, A)$; since $A$ is UI of $C$, there exist functions $f$ and $k$ such that e.g.:

\[
u(C, A) = f_{60}(C) \cdot u(60, A) + k_{60}(C)
\]

We require two equations to find $f_{60}(C)$ and $k_{60}(C)$:

\[
\begin{align*}
u(C, 300) &= f_{60}(C) \cdot u(60, 300) + k_{60}(C) \\
u(C, 600) &= f_{60}(C) \cdot u(60, 600) + k_{60}(C)
\end{align*}
\]

Given that $u(C, 300)$ and $u(C, 600)$ are known functions, we can compute an $f_{60}(c_i)$ and $k_{60}(c_i)$ for each $c_i$. 

\[
\begin{align*}
u(15, 600) &= 0.75 & u(30, 600) &= 0.35 & u(50, 600) &= 0.15 & u(60, 600) &= 0.15 \ 
 u(60, 200) &= 0.25 & u(60, 300) &= 0.20 & u(60, 400) &= 0.15 & u(60, 400) &= 0.00
\end{align*}
\]

In addition, the utility $u(30, 300) = 0.30$ is assessed.

- we then use $u(C, 300)$ to derive all 'vertical' functions from $u(60, A)$; since $A$ is UI of $C$, there exist functions $f$ and $k$ such that e.g.:

\[
u(C, A) = f_{60}(C) \cdot u(60, A) + k_{60}(C)
\]

We require two equations to find $f_{60}(C)$ and $k_{60}(C)$:

\[
\begin{align*}
u(C, 300) &= f_{60}(C) \cdot u(60, 300) + k_{60}(C) \\
u(C, 600) &= f_{60}(C) \cdot u(60, 600) + k_{60}(C)
\end{align*}
\]

Given that $u(C, 300)$ and $u(C, 600)$ are known functions, we can compute an $f_{60}(c_i)$ and $k_{60}(c_i)$ for each $c_i$. 

\[
\begin{align*}
u(15, 600) &= 0.75 & u(30, 600) &= 0.35 & u(50, 600) &= 0.15 & u(60, 600) &= 0.15 \ 
 u(60, 200) &= 0.25 & u(60, 300) &= 0.20 & u(60, 400) &= 0.15 & u(60, 400) &= 0.00
\end{align*}
\]

In addition, the utility $u(30, 300) = 0.30$ is assessed.

- we then use $u(C, 300)$ to derive all 'vertical' functions from $u(60, A)$; since $A$ is UI of $C$, there exist functions $f$ and $k$ such that e.g.:

\[
u(C, A) = f_{60}(C) \cdot u(60, A) + k_{60}(C)
\]

We require two equations to find $f_{60}(C)$ and $k_{60}(C)$:

\[
\begin{align*}
u(C, 300) &= f_{60}(C) \cdot u(60, 300) + k_{60}(C) \\
u(C, 600) &= f_{60}(C) \cdot u(60, 600) + k_{60}(C)
\end{align*}
\]

Given that $u(C, 300)$ and $u(C, 600)$ are known functions, we can compute an $f_{60}(c_i)$ and $k_{60}(c_i)$ for each $c_i$. 

\[
\begin{align*}
u(15, 600) &= 0.75 & u(30, 600) &= 0.35 & u(50, 600) &= 0.15 & u(60, 600) &= 0.15 \ 
 u(60, 200) &= 0.25 & u(60, 300) &= 0.20 & u(60, 400) &= 0.15 & u(60, 400) &= 0.00
\end{align*}
\]
The multilinear utility function — IIa

**THEOREM**

Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n$, $n \geq 2$, and $y_1 \leq \ldots \leq y_m$, $m \geq 2$. If $X$ and $Y$ are mutually utility independent then the two-attribute utility function is multilinear:

$$u(X, Y) = k_X \cdot u_X(X) + k_Y \cdot u_Y(Y) + k_{XY} \cdot u_X(X) \cdot u_Y(Y)$$

where

- $u(X, Y)$ is normalised by $u(x_1, y_1) = 0$ and $u(x_n, y_m) = 1$;
- $u_X(X)$ is a conditional utility function on $X$, normalised by $u_X(x_1) = 0$ and $u_X(x_n) = 1$;
- $u_Y(Y)$ is a conditional utility function on $Y$, normalised by $u_Y(y_1) = 0$ and $u_Y(y_m) = 1$;
- $k_X = u(x_n, y_1) > 0$, $k_Y = u(x_1, y_m) > 0$, $k_{XY} = 1 - k_X - k_Y$ are scaling constants.

Proof of the Theorem:

From the previous lemma we have that for arbitrary values $x_i$ and $y_j$ of $X$ resp. $Y$, we have

$$u(x_i, y_j) = u(x_i, y_1) + u(x_1, y_j) + k \cdot u(x_i, y_1) \cdot u(x_1, y_j)$$

if $u(x_1, y_1) = 0$, $u(x_n, y_1) > 0$ and $u(x_1, y_m) > 0$;

Normalising $u(x_1, Y)$ gives us $u(x_1, Y) = k_Y \cdot u_Y(Y)$ with $k_Y = u(x_1, y_m)$.

Similarly, $u(X, y_1) = k_X \cdot u_X(X)$ with $k_X = u(x_n, y_1)$.

Finally, define $k_{XY} = k \cdot k_X \cdot k_Y$.

The multilinear utility function — IIb

**EXAMPLE**

Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n$, $n \geq 2$, and $y_1 \leq \ldots \leq y_m$, $m \geq 2$.

If $X$ and $Y$ are mutually utility independent then the two-attribute utility function is multilinear:

$$u(X, Y) = u(X, y_1) + u(x_1, Y) + k \cdot u(X, y_1) \cdot u(x_1, Y)$$

where

- $u(x_1, y_1) = 0$, $u(x_n, y_1) > 0$ and $u(x_1, y_m) > 0$;
- $k = \frac{u(x_n, y_m) - u(x_n, y_1) - u(x_1, y_m)}{u(x_1, y_1) \cdot u(x_1, y_m)}$ is a scaling constant.

Proof of the Theorem:

From the previous lemma we have that for arbitrary values $x_i$ and $y_j$ of $X$ resp. $Y$, we have

$$u(x_i, y_j) = u(x_i, y_1) + u(x_1, y_j) + k \cdot u(x_i, y_1) \cdot u(x_1, y_j)$$

if $u(x_1, y_1) = 0$, $u(x_n, y_1) > 0$ and $u(x_1, y_m) > 0$;

Normalising $u(x_1, Y)$ gives us $u(x_1, Y) = k_Y \cdot u_Y(Y)$ with $k_Y = u(x_1, y_m)$.

Similarly, $u(X, y_1) = k_X \cdot u_X(X)$ with $k_X = u(x_n, y_1)$.

Finally, define $k_{XY} = k \cdot k_X \cdot k_Y$.

The multilinear utility function — IIIb

**Proof of Lemma: (sketch):**

Mutual utility independence implies the existence of functions $f > 0$, $g > 0$, $h$, and $k$ such that for arbitrary $x_i$ and $y_i$

(I) $u(X, Y) = g_1(Y) \cdot u(X, y_i) + h_1(Y)$ and

(II) $u(X, Y) = f_1(X) \cdot u(x_i, Y) + k_1(X)$

1. set $y_i$ to $y_1$ and $x_i$ to $x_1$
2. solve (I) for $x_1$ to get $h_1(Y)$ and for $x_n$ to get $g_1(Y)$
3. solve (II) for $y_1$ to get $k_1(X)$ and for $y_m$ to get $f_1(X)$
4. substitute these results in (I) and (II) to (I*) and (II*)
5. now solve (II*) for $x_n$ and fill in this result in (I*) to get the desired result.

The multilinear utility function — IIIa

**LEMMA**

Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n$, $n \geq 2$, and $y_1 \leq \ldots \leq y_m$, $m \geq 2$.

If $X$ and $Y$ are mutually utility independent then the two-attribute utility function is multilinear:

$$u(X, Y) = u(X, y_1) + u(x_1, Y) + k \cdot u(X, y_1) \cdot u(x_1, Y)$$

where

- $u(x_1, y_1) = 0$, $u(x_n, y_1) > 0$ and $u(x_1, y_m) > 0$;
- $k = \frac{u(x_n, y_m) - u(x_n, y_1) - u(x_1, y_m)}{u(x_1, y_1) \cdot u(x_1, y_m)}$ is a scaling constant.
Assessing a multilinear utility function: example

Suppose the City of Utrecht assesses the following conditional utilities for Cost and Acres lost:

\[
\begin{align*}
    u_C(15) &= 1.0 & u_C(30) &= 0.5 & u_C(50) &= 0.2 & u_C(60) &= 0.0 \\
    u_A(200) &= 1.0 & u_A(300) &= 0.8 & u_A(400) &= 0.5 & u_A(600) &= 0.0
\end{align*}
\]

Constants \(k_C = u(15, 600)\) and \(k_A = u(60, 200)\) are assessed:

\[
\begin{align*}
    k_C &= (15, 600) & k_A &= (60, 200)
\end{align*}
\]

The indifference probabilities found: \(k_C = 0.75\) and \(k_A = 0.45\).

So, \(u(C, A) = 0.75 \cdot u_C(C) + 0.45 \cdot u_A(A) - 0.2 \cdot u_C(C) \cdot u_A(A)\)

Interpreting scaling constants (I)

Let \(X\) and \(Y\) be two attributes such that \(Y\) ranges from 0 to 100 and \(u(X, Y) = 0.25 \cdot u_X(X) + 0.75 \cdot u_Y(Y)\).

Suppose that \(u(0, 10) = u(100, 0)\).

Suppose we decide it is sufficient for \(Y\) to range from 0 to 10.

Then rescaling results in:

\[
\begin{align*}
    k'_X &= U'(100, 0) = U'(0, 10) = k'_Y
\end{align*}
\]

Did \(Y\) just turn from three times as important as \(X\) to equally important?

Interpreting scaling constants (II)

Consider a multilinear utility function \(u(X, Y)\) over attributes \(X\) and \(Y\) with values \(x_1 \leq \ldots \leq x_n, n \geq 2\), resp. \(y_1 \leq \ldots \leq y_m, m \geq 2\).

- \(k_{XY}\): consider two lotteries over values \(x_i \leq x_j\) and \(y_k \leq y_l\):

\[
\begin{align*}
    A: & \sim & \frac{1}{2} \quad (x_i, y_k) & \frac{1}{2} \quad (x_j, y_l) \\
    B: & \sim & \frac{1}{2} \quad (x_i, y_k) & \frac{1}{2} \quad (x_j, y_l)
\end{align*}
\]

- \(k_X, k_Y\): if you would rather ‘swing’ \(x_1\) to \(x_n\) than \(y_1\) to \(y_m\), then \(k_X > k_Y\), and vice-versa;

Even a mighty important attribute will have a small scaling constant if its range is relatively small!

The multiplicative utility function

Theorem

Let \(X\) and \(Y\) be two attributes with values \(x_1 \leq \ldots \leq x_n, n \geq 2\), and \(y_1 \leq \ldots \leq y_m, m \geq 2\). If \(X\) and \(Y\) are truly mutually utility independent then the two-attribute utility function is multiplicative:

\[
u(X, Y) = (k \cdot u(X, y_1) + 1) \cdot (k \cdot u(x_1, Y) + 1)\]

where

- \(u(x_1, y_1) = 1, (x_n, y_1) > 1\) and \(u(x_1, y_m) > 1\);

- \(k = \frac{u(x_n, y_m) - u(x_n, y_1) - u(x_1, y_m)}{u(x_1, y_1) \cdot u(x_1, y_m)} > 0\) is a scaling constant.
The multiplicative utility function

Proof of the Theorem:
Mutual utility independence implies that $u(X, Y)$ is multilinear:
$$u(X, Y) = u(X, y_1) + u(x_1, Y) + k \cdot u(X, y_1) \cdot u(x_1, Y)$$
where
- $u(x_1, y_1) = 0$, $u(x_n, y_1) > 0$ and $u(x_1, y_m) > 0$;
- $k = \frac{u(x_n, y_m) - u(x_n, y_1) - u(x_1, y_m)}{u(x_n, y_1) \cdot u(x_1, y_m)}$ is a scaling constant.

Now, $u(X, Y) \sim u^*(X, Y) = k^* \cdot u(X, Y) + 1$, where $k^* = |k| > 0$.
$u^*(X, Y)$ is multiplicative:
$$u^*(X, Y) = k^* \cdot (u(X, y_1) + u(x_1, Y) + k \cdot u(X, y_1) \cdot u(x_1, Y)) + 1$$
$$= (k^* \cdot u(X, y_1) + 1) \cdot (k^* \cdot u(x_1, Y) + 1)$$
$$= u^*(X, y_1) \cdot u^*(x_1, Y)$$
where $u^*(x_1, y_1) = 1$, $u^*(x_n, y_1) > 1$ and $u^*(x_1, y_m) > 1$. $\blacksquare$

The additive utility function — IVa

Corollary
Let $X$ and $Y$ be two attributes with values $x_1, \ldots, x_n$, $n \geq 2$, and $y_1, \ldots, y_m$, $m \geq 2$.

If $X$ and $Y$ are mutually utility independent then
the two-attribute utility function is additive if
$$[0.5, (x_1, y_k); 0.5, (x_i, y_1)] \sim [0.5, (x_1, y_1); 0.5, (x_i, y_k)]$$
for some values $x_1, y_k$ for which
$$(x_1, y_1) \not\sim (x_1, y_k) \text{ and } (x_1, y_1) \not\sim (x_i, y_1)$$

Under the above conditions, we have that
$$u(X, Y) = u(x_1, Y) + u(X, y_1)$$
where $u(x_1, y_1) = 0$, $u(x_n, y_1) > 0$ and $u(x_1, y_m) > 0$.

The additive utility function — IVb

Proof of the Corollary (sketch):
Given mutual utility independence, we have that
$$u(X, Y) = u(x_1, y_1) + u(x_1, Y) + k \cdot u(x_1, y_1) \cdot u(x_1, Y)$$
where $u(x_1, y_1) = 0$, $u(x_n, y_1) > 0$, $u(x_1, y_m) > 0$.
Let $x_i, y_k$ be values such that
$$u(x_1, y_k) + u(x_i, y_1) = u(x_1, y_1) + u(x_i, y_k)$$
that is, $u(x_1, y_k) + u(x_i, y_1) = u(x_i, y_k)$.
Under this constraint, we find from the multilinear form that
$$k \cdot u(x_1, y_k) \cdot u(x_1, y_k) = u(x_1, y_k) + u(x_i, y_1) - u(x_1, y_1) - u(x_1, y_k)$$
$$= 0$$
Since $u(x_i, y_1) \neq u(x_1, y_1)$ and $u(x_1, y_k) \neq u(x_1, y_1)$, $k$ must be zero. $\blacksquare$

An example
Let $X$ and $Y$ be two attributes with values $x_0, x_1, x_2$ and $y_0, y_1, y_2$. Let $u(X, Y)$ be a two-attribute utility function defined as follows:
$$u(x_0, y_0) = 1.0 \quad u(x_0, y_1) = 0.9 \quad u(x_0, y_2) = 0.5$$
$$u(x_1, y_0) = 0.8 \quad u(x_1, y_1) = 0.7 \quad u(x_1, y_2) = 0.3$$
$$u(x_2, y_0) = 0.5 \quad u(x_2, y_1) = 0.4 \quad u(x_2, y_2) = 0.0$$
$X$ and $Y$ are mutually utility independent and for $x_0$ and $y_0$ we have
$$u(x_0, y_0) + u(x_2, y_2) = u(x_0, y_2) + u(x_2, y_0)$$
and
$$u(x_0, y_0) \neq \frac{u(x_0, y_2)}{u(x_2, y_0)}$$
$u(X, Y)$ is therefore additive.
Verifying independences

Preferential independence – the formal definition

Attribute $X$ is preferentially independent of attribute $Y$ [PI] if conditional preferences for values of $X$ given a fixed value for $Y$ do not depend on the particular value for $Y$. More formally,

**Definition**

An attribute $X$ is **preferentially independent** of an attribute $Y$ iff for any consequences $(x_i, y_k)$ and $(x_j, y_k)$ over $X \times Y$ with $Y$ fixed to $y_k$, we have

$$(x_i, y_k) \succeq (x_j, y_k) \implies (x_i, y_l) \succeq (x_j, y_l) \quad \forall y_l \text{ of } Y$$

If $X$ is preferentially independent of $Y$ and $Y$ is preferentially independent of $X$ then $X$ and $Y$ are mutually preferential independent [(M)PI].

Preferential independence – example

Let $X$ and $Y$ have values $x_0, x_1, x_2$ and $y_0, y_1, y_2$. Assume the two-attribute utility function is defined as:

\[
\begin{align*}
    u(x_0, y_0) &= 1.0 & u(x_0, y_1) &= 1.0 & u(x_0, y_2) &= 0.7 \\
    u(x_1, y_0) &= 0.6 & u(x_1, y_1) &= 0.8 & u(x_1, y_2) &= 0.5 \\
    u(x_2, y_0) &= 0.0 & u(x_2, y_1) &= 0.5 & u(x_2, y_2) &= 0.2
\end{align*}
\]

Then we have:

- $X$ is preferentially independent of $Y$:
  \[ x_0 \succeq x_1 \succeq x_2 \] for each value of $Y$;
- $Y$ is not preferentially independent of $X$:
  \[ y_1 \succ y_0 \succ y_2 \text{ for } X = x_1 \]
  \[ y_1 \succ y_2 \succ y_0 \text{ for } X = x_2 \]

Validating preferential independence

Let $X$ and $Y$ be two attributes with values $x_1, \ldots, x_n$, $n \geq 2$, and $y_1, \ldots, y_m$, $m \geq 2$.

Use the following procedure to verify whether $X$ is preferentially independent of $Y$:

1. choose some consequence $(x_j, y_1)$;
2. ask the decision maker for a value $x_i$ for which $(x_i, y_1) \sim (x_j, y_1)$ for some value $x_j$ of $X$ and $y_1$ of $Y$;
3. for a number of values $y \neq y_1$ of $Y$, ask the decision maker whether $(x_i, y) \sim (x_j, y)$ still holds;
4. repeat steps 1, 2 and 3 for different values of $X$.
5. check orientation (no preference reversal?!)
**UI implies PI**

**Proposition**

Consider two attributes \( X \) and \( Y \) with values \( x_1, \ldots, x_n, n \geq 2 \), and \( y_1, \ldots, y_m, m \geq 2 \).

If \( X \) is utility independent of \( Y \) then \( X \) is preferentially independent of \( Y \).

**Proof:** Consider \( x_i, x_j \) and \( y_k \) such that \( (x_i, y_k) \geq u(x_j, y_k) \), that is, \( u(x_i, y_k) - u(x_j, y_k) \geq 0 \).

UI now implies the existence of functions \( g > 0 \) and \( h \) such that for arbitrary \( y_l \):

\[
\begin{align*}
    u(x_i, y_l) &= g_l(y_k) \cdot u(x_i, y_l) = h_l(y_k) \\
    u(x_j, y_l) &= g_l(y_k) \cdot u(x_j, y_l) = h_l(y_k)
\end{align*}
\]

From \( g_l(y_k) > 0 \), we now have \( u(x_i, y_l) - u(x_j, y_l) \geq 0 \) for arbitrary \( y_l \).

---

**PI implies UI ?**

If \( X \) is preferentially independent of \( Y \) then \( X \) not necessarily utility independent of \( Y \).

**Counter example:**

Consider the following utility function:

\[
\begin{align*}
    u(x_0, y_0) &= 1.0 \\
    u(x_1, y_0) &= 0.8 \\
    u(x_2, y_0) &= 0.3 \\
    u(x_0, y_1) &= 0.5 \\
    u(x_1, y_1) &= 0.1 \\
    u(x_2, y_1) &= 0.1
\end{align*}
\]

- \( X \) PI \( Y \) since \( \forall y_i: x_0 \succeq x_1 \succeq x_2 \);
- we have no \( X \) UI \( Y \):

\[
(x_1, y) \sim [p \cdot (x_0, y); (1 - p) \cdot (x_2, y)]
\]

holds for \( p \approx 0.71 \) if \( y \equiv y_0 \), and for \( p = 0 \), if \( y \equiv y_1 \).

---

**Validating utility independence**

If \( X \) PI \( Y \), then \( X \) is also utility independent of \( Y \) if in the following procedure \( x_C \) is equivalent for all values of \( Y \):

1. choose a lottery \([0.5, P; 0.5, Q]\) where \( P = (x_1, y) \) and \( Q = (x_n, y) \) for some value \( y \) of \( Y \) and values \( x_1 \) and \( x_n \) of \( X \);
2. ask the decision maker whether or not he prefers the lottery \([0.5, P; 0.5, Q]\) to a consequence \((x_1, y)\) for some \( x_i, x_1 \leq x_i \leq x_n \);
3. repeat step 2 until you converge to the certainty equivalent \((x_C, y)\) of the lottery;
4. repeat steps 1 – 3 for different values of \( Y \).

Repeat the procedure for different pairs of values for \( X \).

---

**AI implies (M)UI**

**Proposition**

Consider two attributes \( X \) and \( Y \) with values \( x_1, \ldots, x_n, n \geq 2 \), and \( y_1, \ldots, y_m, m \geq 2 \).

If \( X \) and \( Y \) are additive independent then \( X \) and \( Y \) are mutually utility independent.

**Proof (sketch):** We prove \( X \) UI \( Y \); \( Y \) UI \( X \) is analogous.

AI implies \( u(X, y_k) = u(X, y_l) + u(x_i, y_k) \) for arbitrary \( y_k \), where \( u(x_i, y_k) \) is constant w.r.t the value of \( X \).

Continuity implies, for arbitrary \( x_i, x_j \) and \( y_k \) with \( (x_i, y_l) \preceq (x_j, y_l) \preceq (x_k, y_l) \), that \( \exists p \) such that

\[
(l) \quad u(x_j, y_l) = p \cdot u(x_i, y_l) + (1 - p) \cdot u(x_k, y_l).
\]

Substitution of each \( u(X, y_l) \) in (l) with \( u(X, y_k) - u(x_i, y_k) \) gives

\[
u(x_j, y_k) = p \cdot u(x, y_k) + (1 - p) \cdot u(x, y_k) \] for arbitrary \( y_k \).

---
MUI implies AI?

If $X$ and $Y$ are mutually utility independent then $X$ and $Y$ are not necessarily additive independent.

Counter argument:
We have seen that an additional assumption is necessary to conclude additive independence given mutual utility independence (see Additive utility function IVa).

The mentioned corollary can be used to validate AI given MUI; another option is to assume that AI holds...

Validating additive independence – example

Suppose the City of Utrecht assesses the following conditional utilities for Cost and Acres lost:

$$
u_C(15) = 1.0 \quad u_C(30) = 0.5 \quad u_C(50) = 0.2 \quad u_C(60) = 0.0$$

$$u_A(200) = 1.0 \quad u_A(300) = 0.8 \quad u_A(400) = 0.5 \quad u_A(600) = 0.0$$

Suppose $k_C$ is assessed using the following lottery:

$$1.0 \ (15, 600) \sim k_C \ (15, 200) \ (1 - k_C) \ (60, 600)$$

resulting in $k_C = 0.75$. As a consistency check, $k_A$ is assessed as well:

$$1.0 \ (60, 200) \sim k_A \ (15, 200) \ (1 - k_A) \ (60, 600)$$

resulting in $k_A = 0.45$. We now have that $k_C + k_A = 1.2 \neq 1.0$.

Summary

The different functional forms for $u(X, Y)$ are valid under the following conditions:

**additive:**
- iff AI
- iff $X UI Y$ and a lottery assumption for $Y$ (III)
- iff MUI and a lottery assumption for $(x_i, y_k)$ (IV)

**multi-lin.:**
- iff $X UI Y$ and a specific s.e. assumption (I)
- iff MUI (II, III)
- iff AI ($k = 0$) (III)

**multiplic.:**
- iff multi-linear and $k \neq 0$

MAUT with $n = 2$ attributes: isopreference curves
Utility functions with one utility independent attribute

Let \( X \) and \( Y \) be two attributes. Suppose that \( X \) is utility independent of \( Y \), but the reverse does not hold.

Recall that the two-attribute utility function can be specified using three subutility functions:

\[
u(X, Y) = u(x_1, Y) \left[1 - u(X, y_1)\right] + u(x_n, Y) \cdot u(X, y_1)
\]

where \( u(x_1, y_1) = 0 \) and \( u(x_n, y_1) = 1 \).

Other options are:

- replacing one or two subutility functions with one or two, respectively, isopreference curves.

Isopreference curves – definition

An isopreference curve describes a set of consequences that are equally desirable to the decision maker.

**Definition**

Consider two (sets of) attributes \( X \) and \( Y \) and a partial function \( i : X \rightarrow Y \) such that for arbitrary values \( x_i \neq x_k \) of \( X \) and \( y_j, y_l \) of \( Y \), we have that

\[
i(x_i) = y_j \text{ and } i(x_k) = y_l \iff (x_i, y_j) \sim (x_k, y_l)
\]

An isopreference curve is an interpolant \( r^Y(X) \) of \( i(X) \).

Note that for any two points \((x_i, y_j)\) and \((x_k, y_l)\) on the isopreference curve, we thus have that

\[
u(x_i, y_j) = u(x_k, y_l) = c \text{ for some constant } c.
\]

As a result, \( u(X, r^Y(X)) = c \) is a constant subutility function, defined on all values of \( X \).

**Isopreference curves – details (I)**

Here we have for all values \( x \) of \( X \) that

\[
(x, r^1_Y(x)) \sim (x_1, y_i), \quad \text{ that is: } \quad u(x, r^1_Y(x)) = u(x_1, y_i)
\]

\[
(x, r^2_Y(x)) \sim (x_1, y_j), \quad \text{ that is: } \quad u(x, r^2_Y(x)) = u(x_1, y_j)
\]
An example

Jenny wants to treat her friends to some cookies and candy. Let $x_i$ denote $i$ cookies and $y_j$ indicate $j$ pieces of candy.

Jenny is indifferent between $(x_2, y_1), (x_6, y_6)$, and $(x_8, y_4)$. Also, Jenny is indifferent between $(x_1, y_6), (x_3, y_5)$, and $(x_7, y_3)$.

Assessment of one utility for a single point on the $ar{r}^X_1(X)$ curve gives the utility of all points $(x, \bar{r}^X_1(x))$.

We have for example that $\bar{r}^Y_1(x_2) = y_{10}$ and $\bar{r}^Y_2(x_3) = y_5$, but also $\bar{r}^Y_1(x_4) = y_8$ and $\bar{r}^Y_2(x_4) = y_4$.

Substitution of $u(\bar{r}^X(Y), Y)$ for $u(x_n, Y)$ (a)

**Corollary**

Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n$, $n \geq 2$, and $y_1 \leq \ldots \leq y_m$, $m \geq 2$.

If $X$ is utility independent of $Y$ then

$$u(X, Y) = u(x_1, Y) + \left[ \frac{u(x_k, y_1) - u(x_1, Y)}{u(\bar{r}^X(Y), y_1)} \right] \cdot u(X, y_1)$$

where

- $u(x_1, y_1) = 0$
- $\bar{r}^X(Y)$ is defined such that $(\bar{r}^X(Y), Y) \sim (x_k, y_1)$ for an arbitrary $x_k \neq x_1$.

One isopreference curve for one subutility function

Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n$, $n \geq 2$, and $y_1 \leq \ldots \leq y_m$, $m \geq 2$.

Recall that if $X$ is utility independent of $Y$ then

$$u(X, Y) = u(x_1, Y) [1 - u(X, y_1)] + u(x_n, Y) \cdot u(X, y_1)$$

Replace subutility function $u(x_n, Y)$ over all $y_i$ by subutility function $u(\bar{r}^X(Y), Y)$ over all $y_i$:

Replace subutility function $u(X, y_i)$ over all $x_i$ by subutility function $u(X, \bar{r}^Y(X))$ over all $x_i$:

Substitution of $u(\bar{r}^X(Y), Y)$ for $u(x_n, Y)$ (b)

**Proof (sketch):**

Utility independence implies the existence of functions $g > 0, h$ such that

(1) $u(X, Y) = g(Y) \cdot u(X, y_1) + h(Y) \forall y_i$

1. set $y_i$ to $y_1$;
2. solve (1) for $x_1$ to get $h(Y)$, using $u(x_1, y_1) = 0$;
3. let $x_k$ be the point where $\bar{r}^X(Y)$ intersects the line $(X, y_1)$, then $u(\bar{r}^X(Y), Y) = u(x_k, y_1)$; use this in solving (1) for $\bar{r}^X(Y)$ to get $g(Y)$;
4. substitute these results in (1) to get the desired result. ■
An example

Suppose the City of Utrecht assesses the following utilities for Cost and Acres lost:

\[ u(15, 600) = 0.75 \quad u(60, 100) = 0.25 \]
\[ u(30, 600) = 0.50 \quad u(60, 200) = 0.20 \]
\[ u(45, 600) = 0.20 \quad u(60, 300) = 0.15 \]
\[ u(50, 600) = 0.10 \quad u(60, 400) = 0.10 \]
\[ u(60, 600) = 0.00 \quad u(60, 600) = 0.00 \]

In addition, the City indicates the indifferences
\[ (50, 200) \sim (45, 300) \sim (40, 400) \]
resulting in an isopreference curve \( \bar{r}^C(A) \) with \( u(\bar{r}^C(A), A) = 0.40 \) and e.g. \( \bar{r}^C(600) = 35 \) (does not follow from above).

If Cost is utility independent of Acres lost, then
\[ u(C, A) = u(60, A) + \left[ \frac{u(35, 600) - u(60, A)}{u(\bar{r}^C(600), 600)} \right] \cdot u(C, 600) \]

Substitution of \( u(X, \bar{r}^Y(X)) \) for \( u(X, y_1) \) (b)

Proof (sketch):
Utility independence implies the existence of functions \( g > 0, h \) such that
\[ (I) \quad u(X, Y) = g(Y) \cdot u(X, y_1) + h(Y) \forall y_1 \]

Now let \( y_k \) be the point where \( \bar{r}^Y(X) \) intersects the line \( (x_1, Y) \), then \( u(X, \bar{r}^Y(X)) = u(x_1, y_k) \).

1. set \( y_k = y_1 \);
2. solve (I) for \( x_1 \) to get \( h_k(Y) \), using \( u(x_1, y_k) = 0 \)
3. solve (I) for \( x_n \) to get \( g_k(Y) \);
4. solve (I) for \( r^Y(X) \) to get \( u(X, y_k) \).
5. substitute these results in (I) to get the desired result.

Substitution of \( u(X, \bar{r}^Y(X)) \) for \( u(X, y_1) \) (a)

**Corollary**

Let \( X \) and \( Y \) be two attributes with values \( x_1 \leq \ldots \leq x_n, n \geq 1 \), and \( y_1 \leq \ldots \leq y_m, m \geq 1 \).

If \( X \) is utility independent of \( Y \) then
\[ u(X, Y) = \frac{u(x_1, Y) \cdot u(x_n, \bar{r}^Y(X)) - u(x_n, Y) \cdot u(x_1, \bar{r}^Y(X))}{u(x_1, \bar{r}^Y(X)) - u(x_1, \bar{r}^Y(X))} \]

where
- \( \bar{r}^Y(X) \) is defined such that \( (X, \bar{r}^Y(X)) \sim (x_1, y_k) \) for an \( y_k \) with:
- \( u(x_1, y_k) = 0, (x_1 \neq x_n) \)

An example

Suppose the City of Utrecht assesses the following utilities for Cost and Acres lost:

\[ u(15, 200) = 1.00 \quad u(60, 200) = 0.20 \]
\[ u(15, 300) = 0.90 \quad u(60, 300) = 0.15 \]
\[ u(15, 400) = 0.80 \quad u(60, 400) = 0.10 \]
\[ u(15, 600) = 0.75 \quad u(60, 600) = 0.00 \]
\[ u(15, 900) = 0.50 \quad u(60, 900) = -0.15 \]

In addition, the City indicates the indifferences
\[ (15, 1500) \sim (50, 700) \sim (60, 600) \]
resulting in an isopreference curve \( \bar{r}^A(C) \) with
\[ u(C, \bar{r}^A(C)) = 0.00 \] and e.g. \( \bar{r}^A(30) = 900 \).

If Cost is utility independent of Acres lost, then
\[ u(C, A) = \frac{u(60, A) \cdot u(15, \bar{r}^A(C)) - u(15, A) \cdot u(60, \bar{r}^A(C))}{u(15, \bar{r}^A(C)) - u(60, \bar{r}^A(C))} \]
Use of two isopreference curves

Recall that if $X \succ_{UI} Y$ then

$$u(X, Y) = u(x_1, Y) \left[1 - u(X, y_1)\right] + u(x_n, Y) \cdot u(X, y_1)$$

Repeating $u(x_1, Y)$ by $u(\overline{r}_1^X(Y), Y),$ and $u(x_n, Y)$ by $u(\overline{r}_2^X(Y), Y)$:

Is there another possibility with two isopreference curves?

Substitution of $u(\overline{r}_1^X(Y), Y)$ and $u(\overline{r}_2^X(Y), Y)$ for $u(x_1, Y)$ and $u(x_n, Y)$ (a)

**Proof (sketch):**

Utility independence implies $\exists g > 0, h$ such that

$$u(X, Y) = g(Y) \cdot u(X, y) + h(Y) \forall y$$

Let $x_k, x_1$ be the intersection points for $\overline{r}_1^X(Y), \overline{r}_2^X(Y),$ resp., with $(X, y_1)$.

1. set $y_i$ to $y_1$;
2. solve (I) for $x_k$ and $\overline{r}_1^X(Y)$ to get $h(Y)$ and $g(Y)$, using $u(\overline{r}_1^X(Y), Y) = u(x_k, y_1) = 0$;
3. solve (I) for $\overline{r}_2^X(Y)$ to get $u(x_k, Y)$, using $u(\overline{r}_2^X(Y), Y) = u(x_1, y_1) = 1$;
4. substitute these results in (I).

Substitution of $u(\overline{r}_1^X(Y), Y)$ and $u(\overline{r}_2^X(Y), Y)$ for $u(x_1, Y)$ and $u(x_n, Y)$ (b)

Let $X$ and $Y$ be two attributes with values $x_1 \leq \ldots \leq x_n, n \geq 2$, and $y_1 \leq \ldots \leq y_m, m \geq 2$.

If $X$ is utility independent of $Y$ then

$$u(X, Y) = \frac{u(X, y_1) - u(\overline{r}_1^X(Y), y_1)}{u(\overline{r}_2^X(Y), y_1) - u(\overline{r}_1^X(Y), y_1)}$$

where

- $u(X, Y)$ is normalised by $u(x_k, y_1) = 0$ and $u(x_1, y_1) = 1$;
- $\overline{r}_1^X(Y)$ is defined such that $(\overline{r}_1^X(Y), Y) \sim (x_k, y_1)$;
- $\overline{r}_2^X(Y)$ is defined such that $(\overline{r}_2^X(Y), Y) \sim (x_1, y_1)$

An example

Suppose the City of Utrecht assesses the following utilities for Cost and Acres lost:

- $u(5, 600) = 1.00$  $u(50, 600) = 0.10$
- $u(12, 600) = 0.70$  $u(60, 600) = 0.00$
- $u(15, 600) = 0.75$  $u(75, 600) = -0.10$
- $u(30, 600) = 0.50$

In addition, the City indicates the indifferences

- $(80, 200) \sim (75, 300) \sim (70, 400) \sim (60, 600)$

resulting in isopreference curve $\overline{r}_1^X(A)$ with $u(\overline{r}_1^X(A), A) = 0.00$, and the indifferences

- $(15, 200) \sim (12, 300) \sim (10, 400) \sim (5, 600)$

resulting in isopreference curve $\overline{r}_2^X(A)$ with $u(\overline{r}_2^X(A), A) = 1.00$.

If Cost is utility independent of Acres lost, then

$$u(C, A) = \frac{u(C, 600) - u(\overline{r}_1^X(A), 600)}{u(\overline{r}_2^X(A), 600) - u(\overline{r}_1^X(A), 600)}$$