Visual interpretation with normal approximation

$H_0$ is true: $\alpha = 0.05$ (Type I error rate)

$H_1$ is true: $p = 0.06$

Fail to reject $H_0$

$\beta = 0.6468$ (Type II error rate)

Accept $H_1$

$x = 0.05$
Visual interpretation with normal approximation

$H_0$ is true:

$H_0$ is true: $\alpha = 0.05$ (Type I error rate)

25 33

$H_1$ is true: $p = 0.08$

Fail to reject $H_0$

$\beta = 0.0936$ (Type II error rate)

Accept $H_1$

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$H_0$ is true: $\alpha = 0.05$ (Type I error rate)

$H_1$ is true: $p = 0.10$

$\beta = 0.0036$ (Type II error rate)
Summary

Properties of hypothesis testing

1. $\alpha$ and $\beta$ are related; decreasing one generally increases the other.
2. $\alpha$ can be set to a desired value by adjusting the critical value. Typically, $\alpha$ is set at 0.05 or 0.01.
3. Increasing $n$ decreases both $\alpha$ and $\beta$.
4. $\beta$ decreases as the distance between the true value and hypothesized value ($H_1$) increases.
In our examples so far we have considered:

- $H_0: \theta = \theta_0$
- $H_1: \theta > \theta_0$.

This is a one-tailed test with the critical region in the right-tail of the test statistic $X$. 
Another one-tailed test could have the form,

- $H_0: \theta = \theta_0$
- $H_1: \theta < \theta_0$,

in which the critical region is in the left-tail.
In a two-tailed test check for differences:

- $H_0$: $\theta = \theta_0$
- $H_1$: $\theta \neq \theta_0$, 

![Diagram showing two-tailed test](attachment:diagram.png)
Two cases:

Variance known

Variance unknown
Two cases:

✓ Variance known

✗ Variance unknown
Consider a production line of resistors that are supposed to be 100 Ohms. Assume $\sigma = 8$. So, the hypotheses are:

- $H_0$: $\mu = 100$
- $H_1$: $\mu \neq 100$,

Let $\bar{X}$ be the sample mean for a sample of size $n = 100$.

<table>
<thead>
<tr>
<th>Reject $H_0$</th>
<th>Do not reject $H_0$</th>
<th>Reject $H_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>98</td>
<td>102</td>
<td></td>
</tr>
</tbody>
</table>

In this case the test statistic is the sample mean because this is a continuous random variable.
We know the sampling distribution of $\bar{X}$ is a normal distribution with mean $\mu$ and standard deviation $\sigma/\sqrt{n} = 0.8$ due to the central limit theorem.
Tests concerning sample mean
(variance known)

As in the previous example, we are often interested in testing

- $H_0$: $\mu = \mu_0$
- $H_1$: $\mu \neq \mu_0$,

based on the sample mean $\bar{X}$ from samples $X_1, X_2, \ldots, X_n$, with known population variance $\sigma^2$.

Under $H_0 : \mu = \mu_0$, the probability of a type I error is computed using the sampling distribution of $\bar{X}$, which, due to the central limit theorem, is normal distributed with mean $\mu$ and standard deviation $\sigma/\sqrt{n}$. 
From confidence intervals we know that

$$\Pr \left( -z_{\alpha/2} < \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} < z_{\alpha/2} \right) = 1 - \alpha$$
Therefore, to design a test at the level of significance $\alpha$ we choose the critical values $a$ and $b$ as

\[
a = \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}
\]

\[
b = \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}},
\]

then we collect the sample, compute the sample mean $\bar{X}$ and reject $H_0$ if $\bar{X} < a$ or $\bar{X} > b$. 
Tests concerning sample mean (cont.)
(variance known)

Steps in hypothesis testing

1. State the null and alternate hypothesis
2. Choose a significance level \( \alpha \)
3. Choose the test statistic and establish the critical region
4. Collect the sample and compute the test statistic. If the test statistic is in the critical region, reject \( H_0 \). Otherwise, do not reject \( H_0 \).
Example
A batch of 100 resistors have an average of 102 Ohms.
Assuming a population standard deviation of 8 Ohms, test
whether the population mean is 100 Ohms at a significance
level of $\alpha = 0.05$.

Step 1:

$H_0 : \mu = 100$

$H_1 : \mu \neq 100$,

Note: Unless stated otherwise, we use a two-tailed test.

Step 2: $\alpha = 0.05$
Example continued

Step 3: In this case, the test statistic is specified by the problem to be the sample mean $\bar{X}$. Reject $H_0$ if $\bar{X} < a$ or $\bar{X} > b$, with

$$a = \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = \mu_0 - z_{0.025} \frac{\sigma}{\sqrt{100}}$$

$$= 100 - 1.96 \frac{8}{10} = 98.432$$

$$b = \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 100 + 1.96 \frac{8}{10} = 101.568.$$ 

Step 4: We are told that the test statistic on a sample is $\bar{X} = 102 > b$. Therefore, reject $H_0$. 

Tests concerning sample mean (cont.)
(variance known)