The Stable Marriage Problem

Algorithms and Networks

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The stable marriage problem

- Story: there are *n* men and *n* women, which are unmarried. Each has a preference list on the persons of the opposite sex
- Does there exist and can we find a stable matching (stable marriage): a matching of men and women, such that there is no pair of a man and a woman who both prefer each other above their partner in the matching?

Application

- Origin: assignment of medical students to hospitals (for internships)
  - Students list hospitals in order of preference
  - Hospitals list students in order of preference

Example

- Arie: Betty Ann Cindy
- Bert: Ann Cindy Betty
- Carl: Ann Cindy Betty
- Ann: Bert Arie Carl
- Betty: Arie Carl Bert
- Cindy: Bert Arie Carl

- Stable matching: (Arie,Betty), (Bert,Ann), (Carl,Cindy)
- Matching (Arie,Ann), (Bert,Betty), (Carl, Cindy) is not stable, e.g., Arie and Betty prefer each other above given partner
- *Blocking pair*

Remark

- "Local search" approach does not need to terminate

  SOAP-SERIES-ALGORITHM

  While there is a blocking pair

  Do Switch the blocking pair

  – Can go on for ever!

  – So, we need something else…

Result

- Gale/Stanley algorithm: finds always a stable matching
  - Input: list of men, women, and their preference list
  - Output: stable matching

The algorithm

- Fix some ordering on the men
- Repeat until everyone is matched
  - Let X be the first unmatched man in the ordering
  - Find woman Y such that Y is the most desirable woman in X's list such that Y is unmatched, or Y is currently matched to a Z and X is more preferable to Y than Z.
  - Match X and Y; possible this turns Z to be unmatched

Questions:
Does this terminate? How fast?
Does this give a stable matching?

Termination and number of steps

- Once a woman is matched, she stays matched (her partner can change).
- When the partner of a woman changes, this is to a more preferable partner for her: at most $n - 1$ times.
- Every step, either an unmatched woman becomes matched, or a matched woman changes partner: at most n^2 steps.

Stability of final matching

- Suppose final matching is not stable.
- Take:
  - M*x* is matched to W*x*,
  - M*y* is matched to W*y*,
  - M*x* prefers W*y* to W*x*,
  - W*y* prefers M*x* to M*y*.
- When W*y* is before W*x* in the preference list of M*x*, but M*x* is not matched to W*y*. Two cases:
  - When M*x* considers W*y*, she has a partner M*z* preferable to M*x*: M*z* is also preferable to M*y,* but in the algorithm woman can get only more preferable partners, contradiction.
  - When M*x* considers W*y,* she is free, but M*x* is later replaced by someone preferable to M*x*. Again, W*y* can never end up with M*y*.

Comments

- A stable matching exists and can be found in polynomial time
- Consider the greedy algorithm:
  - Start with any matching, and make switches when a pair prefers each other to their current partner
  - This algorithm does not need to terminate
- Controversy: the algorithm is better for the men: hospitals in the application

Man optimal stable matchings

- All possible executions of the Gale-Shapley algorithm give the same stable matching.
- In this matching, the men have the best partner they can have in any stable matching.
- In this matching, the women have the worst partner they can have in any stable matching.

Proof

- Suppose the algorithm gives matching M.
- Suppose there is a stable matching M’ with man *m* matched to *w’* in M’, and to *w* in M, with *m* preferring *w’* over *w*.
- Look at run of algorithm that produces M. *w’* has rejected *m* at some point.
- Of all such *m, w* and *w’,* take a triple such that the rejection of *m* by *w’* happens first.
- Suppose *w’* prefers *m’* to *m*, as reason for the rejection.
- *m’* must prefer *w’* to his partner in M’: see next slide
- Thus *m’, w’* is a *blocking pair* in M’: M’ not stable; contradiction.
- So, all men are matched to the woman that appears in a stable matching that they prefer most.
  - Unique solution

m' Prefers *w'*

- *m'* is matched in M' with *w''*
- If *m'* prefers *w''* to *w'*:
  - In execution of algorithm, we have
    - At some point *w''* must reject *m'* as later *m'* is matched with *w'* (while *w'* rejects *m* at that step).
    - This is earlier than the rejection of *w'* of *m*
  - Now *m', w'* and *w''* form an earlier choice for the triple.

Stable roommates

- Variant of problem with boys that must share two-person rooms (US campus)
- Each has preference list
- Stable marriage problem is special case

Not always a stable matching for the stable roommates

- Consider the following instance:
  - Person Arie: Carl Bert Dirk
  - Person Bert: Arie Carl Dirk
  - Person Carl: Bert Arie Dirk
  - Person Dirk: no difference
- Each matching is unstable e.g., (Arie,Bert) (Carl,Dirk) has {Carl,Arie} as blocking pair

Testing stable roommates

- Complicated algorithm
- Uses $O(n^2)$ time

Comments

- Much further work has been done, e.g.:
  - Random / Fair stable matchings
  - Many variants of stable matching are also solvable (indifferences, groups, forbidden pairs, …)
  - What happens if some participants lie about their preferences?
  - Stable roommates with indifferences: NP-complete