The Stable Marriage Problem

Algorithms and Networks
The stable marriage problem

- Story: there are $n$ men and $n$ women, which are unmarried. Each has a preference list on the persons of the opposite sex.

- Does there exist and can we find a stable matching (stable marriage): a matching of men and women, such that there is no pair of a man and a woman who both prefer each other above their partner in the matching?
Application

- Origin: assignment of medical students to hospitals (for internships)
  - Students list hospitals in order of preference
  - Hospitals list students in order of preference
Example

- Arie: Betty Ann Cindy
- Bert: Ann Cindy Betty
- Carl: Ann Cindy Betty
- Ann: Bert Arie Carl
- Betty: Arie Carl Bert
- Cindy: Bert Arie Carl

- Stable matching: (Arie, Betty), (Bert, Ann), (Carl, Cindy)
- Matching (Arie, Ann), (Bert, Betty), (Carl, Cindy) is not stable, e.g., Arie and Betty prefer each other above given partner
- **Blocking pair**
Remark

- “Local search” approach does not need to terminate

SOAP-SERIES-ALGORITHM

While there is a blocking pair

Do Switch the blocking pair

- Can go on for ever!
- So, we need something else…
Result

- **Gale/Stanley algorithm**: finds always a stable matching
  - Input: list of men, women, and their preference list
  - Output: stable matching
The algorithm

• Fix some ordering on the men
• Repeat until everyone is matched
  – Let X be the first unmatched man in the ordering
  – Find woman Y such that Y is the most desirable woman in X’s list such that Y is unmatched, or Y is currently matched to a Z and X is more preferable to Y than Z.
  – Match X and Y; possible this turns Z to be unmatched

Questions:
Does this terminate? How fast?
Does this give a stable matching?
Termination and number of steps

- Once a woman is matched, she stays matched (her partner can change).
- When the partner of a woman changes, this is to a more preferable partner for her: at most $n - 1$ times.
- Every step, either an unmatched woman becomes matched, or a matched woman changes partner: at most $n^2$ steps.
Stability of final matching

- Suppose final matching is not stable.

- Take:
  - $M_x$ is matched to $W_x$,
  - $M_y$ is matched to $W_y$,
  - $M_x$ prefers $W_y$ to $W_x$,
  - $W_y$ prefers $M_x$ to $M_y$.

- When $W_y$ is before $W_x$ in the preference list of $M_x$, but $M_x$ is not matched to $W_y$. Two cases:
  - When $M_x$ considers $W_y$, she has a partner $M_z$ preferable to $M_x$: $M_z$ is also preferable to $M_y$, but in the algorithm woman can get only more preferable partners, contradiction.
  - When $M_x$ considers $W_y$, she is free, but $M_x$ is later replaced by someone preferable to $M_x$. Again, $W_y$ can never end up with $M_y$. 

Comments

• A stable matching exists and can be found in polynomial time

• Consider the greedy algorithm:
  – Start with any matching, and make switches when a pair prefers each other to their current partner
  – This algorithm does not need to terminate

• Controversy: the algorithm is better for the men: hospitals in the application
Man optimal stable matchings

- All possible executions of the Gale-Shapley algorithm give the same stable matching.
- In this matching, the men have the best partner they can have in any stable matching.
- In this matching, the women have the worst partner they can have in any stable matching.
Proof

- Suppose the algorithm gives matching $M$.
- Suppose there is a stable matching $M'$ with man $m$ matched to $w'$ in $M'$, and to $w$ in $M$, with $m$ preferring $w'$ over $w$.
- Look at run of algorithm that produces $M$. $w'$ has rejected $m$ at some point.
- Of all such $m$, $w$ and $w'$, take a triple such that the rejection of $m$ by $w'$ happens first.
- Suppose $w'$ prefers $m'$ to $m$, as reason for the rejection.
- $m'$ must prefer $w'$ to his partner in $M'$: see next slide
- Thus $m'$, $w'$ is a blocking pair in $M'$: $M'$ not stable; contradiction.
- So, all men are matched to the woman that appears in a stable matching that they prefer most.
  - Unique solution
$m'$ Prefers $w'$

- $m'$ is matched in $M'$ with $w''$
- If $m'$ prefers $w''$ to $w'$:
  - In execution of algorithm, we have
    - At some point $w''$ must reject $m'$ as later $m'$ is matched with $w'$ (while $w'$ rejects $m$ at that step).
    - This is earlier than the rejection of $w'$ of $m$
  - Now $m'$, $w'$ and $w''$ form an earlier choice for the triple.
Stable roommates

- Variant of problem with boys that must share two-person rooms (US campus)
- Each has preference list
- Stable marriage problem is special case
Not always a stable matching for the stable roommates

• Consider the following instance:
  – Person Arie: Carl Bert Dirk
  – Person Bert: Arie Carl Dirk
  – Person Carl: Bert Arie Dirk
  – Person Dirk: no difference

• Each matching is unstable e.g., (Arie,Bert) (Carl,Dirk) has \{Carl,Arie\} as blocking pair
Testing stable roommates

- Complicated algorithm
- Uses $O(n^2)$ time
Comments

• Much further work has been done, e.g.:
  – Random / Fair stable matchings
  – Many variants of stable matching are also solvable (indifferences, groups, forbidden pairs, …)
  – What happens if some participants lie about their preferences?
  – Stable roommates with indifferences: NP-complete