Minimum Cost Flow

Algorithms and Networks

Universiteit Utrecht

This lecture

- The minimum cost flow problem: statement and applications
- The cycle cancelling algorithm
- A polynomial time variant of cycle cancelling
- The successive shortest paths algorithm

Minimum Cost Flow I

- Edges have
  - Capacity c(u,v): bound on amount of flow that can go through the edge
  - Cost: cost(u,v): cost that must be paid per unit of flow that goes through the edge.
- Cost of flow f:
  - Sum over all (u,v) of: f (u,v) * cost(u,v).

Minimum cost flow problem

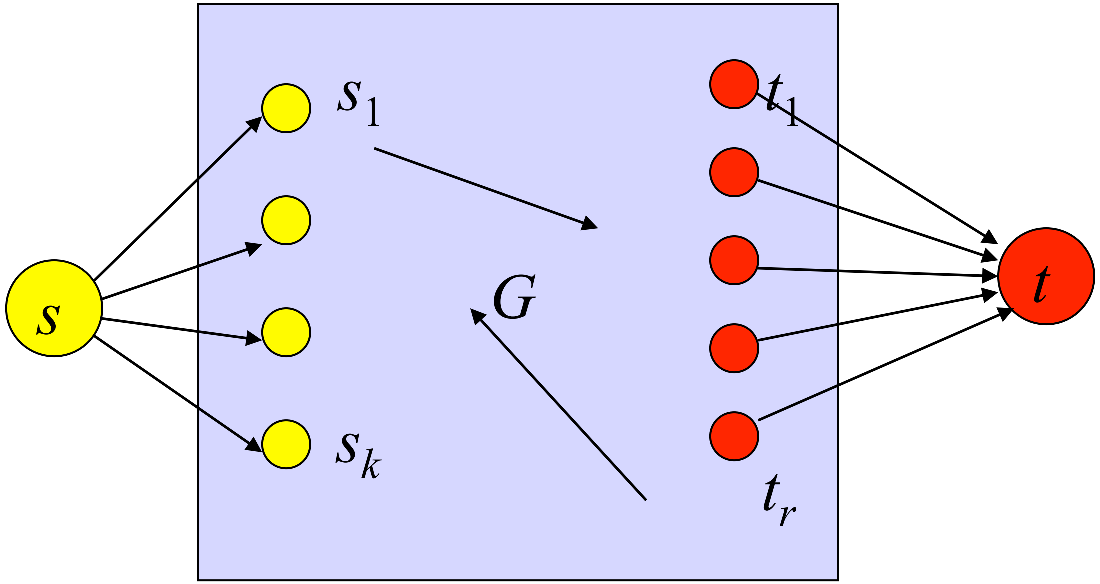
- **Given**: Network G, c, cost, *s*, *t*, and a target flow value *r*.
- **Question**: Find a flow from *s* to *t* with value *r*, with minimum cost.

Unbounded capacities

- Some edges may have unbounded capacities
  - If there is a cycle of negative cost with only edges with unbounded capacity:
    - Arbitrary small cost (degenerate case)
  - Otherwise: simple transformation to bounded capacities
    - E.g., set each unbounded capacity to sum of all bounded capacities

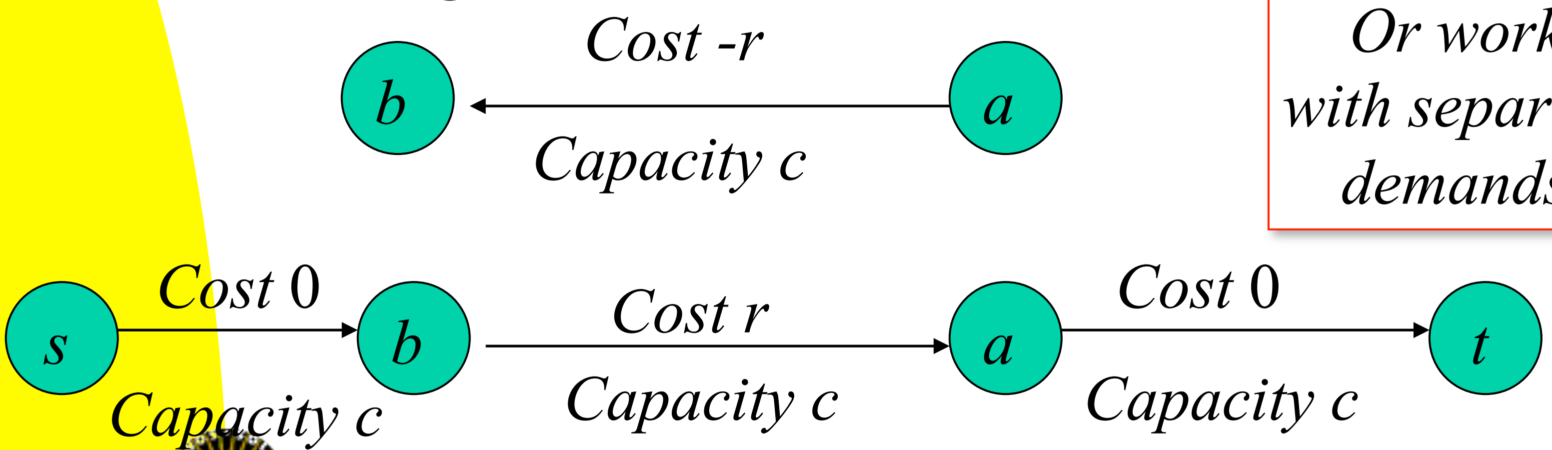
Separate demands

- Similar to max-flow with multiple sources and multiple sinks
- Or: work with "demand", which can be positive or negative



Nonnegative arc costs

- We may assume all costs are nonnegative.
- In case of negative costs: assume bounded capacity. Modify network to equivalent one with nonnegative costs:

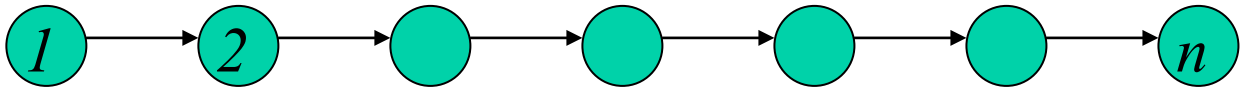


Or work with separate demands

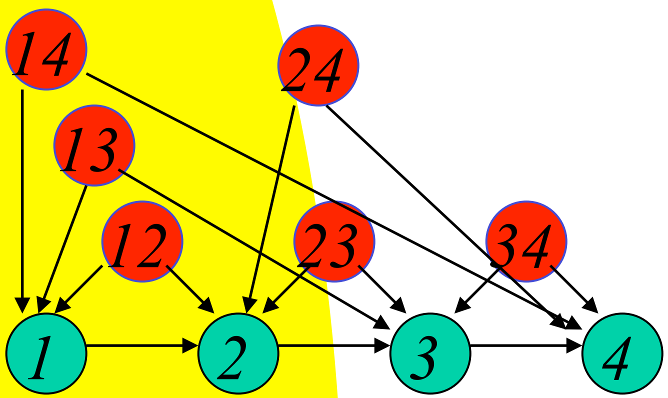
Applications

- Transport problems
- Minimum cost matchings
- Reconstruction of Left Ventricle from X-ray projections
  - Image: 2d bit array; known are sums of columns, rows; probabilities for each bit
  - Look for image with correct row and column sums of maximum probability
  - Can be modelled as minimum cost flow problem

Application: Optimal loading of hopping airplane



b_{ij} weight units (or passengers) can be transported from i to j
Each gives a profit of f_{ij}
Plane can never carry more than p units
How much units do we transport of each type for maximum profit?



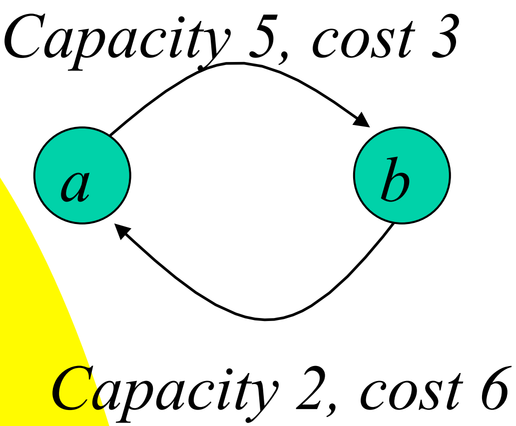
Capacity $i \rightarrow j$: p
Node ij has supply b_{ij}
Cost from ij to i: $- f_{ij}$
Node i has demand sum over all b_{ji}

All other arcs infinite cap.
All other costs 0

Residual network

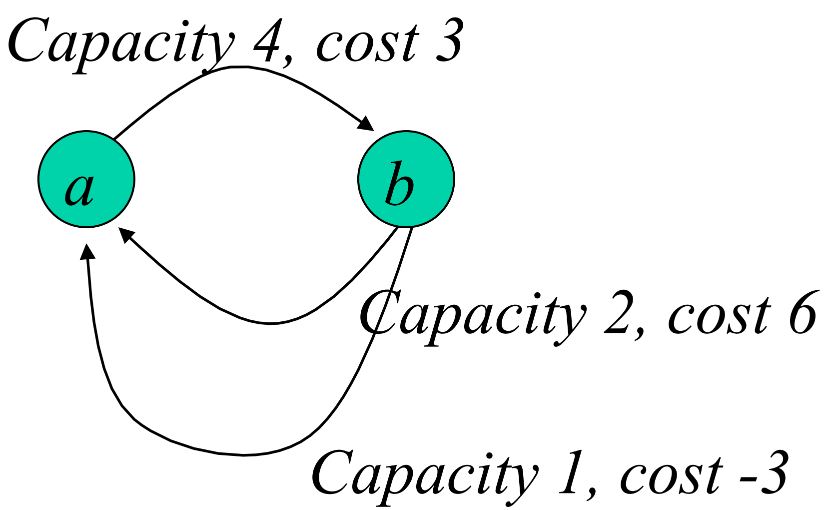
- Capacities as for maximum flow algorithms.
- If $f(u,v)>0$, then $\text{cost}_f(u,v) = \text{cost}(u,v)$, and $\text{cost}_f(v,u) = -\text{cost}(u,v)$.

Example



Suppose we send 1 flow from a to b

In G_f:



Cycle cancelling algorithm

- Make a feasible flow *f* in the network
- **while** G_f has a negative cycle **do**
  - Find a negative cycle C in G_f
  - Let D be the minimum residual capacity c_f of an edge on C
  - Add D units of flow to each edge on C: this is a new feasible flow of smaller cost
- Output *f*.

Cycle cancelling algorithm is correct

- Theorem: a flow f has minimum costs, if and only if G_f has no negative cycle.
  - If G has negative cycle, then we can improve f to one with smaller cost.
  - Suppose f is a flow, and f' is an optimal flow. $f' - f$ is a circulation in G_f, hence a linear combination of cycles, and if f is not optimal, then the total cost of these cycles is negative, so there is a negative cycle in this set: it is a cycle in G_f.

More on the cycle cancelling algorithm

- No guarantee that it uses polynomial time.
- Corollary: if all costs, capacities, and target flow value are integral, then there is an optimal *integer* minimum cost flow.
  - The cycle cancelling algorithm finds an integer flow in this case.
- Variant: using always the minimum mean cost cycle gives a polynomial time algorithm!

Minimum mean-cost circulation algorithm

- Cycle cancelling algorithm but find always the minimum mean cost cycle and use that.
  - O(nm) time to find the cycle.
  - A theorem shows that O($nm^2 \log^2 n$) iterations are sufficient.
  - O($n^2 m^3 \log^2 n$) algorithm.

Successive shortest paths

- Start with flow f with $f(u,v)$=0 for all u,v.
- **repeat until** value(f) = r
  - Find the shortest path P in G_f from s to t
  - Let q be the minimum residual capacity of an edge on P.
  - Send min(q,r – value(f)) additional units of flow across P.

On the successive shortest paths algorithm

- May use exponential running time
- Assume G has no negative edge costs.
- Gives optimal answer.
  - Invariant: f has minimum cost among all flows with value *value*(f).

- Suppose we obtain f' from f by sending across P.
- Let f'' be a minimum cost flow with same value as f'.
- Write $f'' - f$ as weighted sum of paths from s to t in G_f and circuits in G_f. Argue that cost($f' - f$) ≤ cost($f'' - f$), using that:
  - P is shortest path
  - Circuits have non-negative costs, by optimality of f.

Finally

- More efficient algorithms exist
  - Some use scaling techniques:
    - Scaling on capacities
    - Scaling on costs