Exercise set VI, Algorithms and Networks 2017-2018

Please hand in your solutions either by email to H.L.Bodlaender@uu.nl or on paper during the course or in the mailbox of Hans Bodlaender in the coffeeroom of the 5th floor of the Buys Ballot Gebouw. Note that you may work alone or work in pairs, see the website for further details. The deadline for this exercise set is Monday, January 22nd 2018, 23:59.

In case you hand in your work by email:

- Have [AN6] in the header of the email.
- Attach ONE pdf-file to your email. (If you use Word, note that Word can output the text as pdf-file.)
- Make sure that your name(s) and student number(s) is/are in the pdf-file.

You can write either in English or in Dutch. Your work should be well phrased, readable, understandable, and of course, correct. It is not your teachers job to figure out what you might mean with what you have written. As such, both messy work and work that is difficult to understand can be graded with a 0.

1. Another approximation algorithm for vertex cover (2.5 points)

We consider the vertex cover problem. In this exercise, we look to another approximation algorithm for this problem. This algorithm uses Depth First Search.

The algorithm works as follows. Suppose G is connected. Run the Depth First Search algorithm, and obtain the DFS-tree T. (This is the tree formed by the edges through which the DFS graph search algorithm goes into recursion to new unvisited vertices) Let C be the set of all vertices that have at least one child in T, i.e., C is the set of all vertices that are not a leaf in T. Now, Output C.

a) Prove that C is a vertex cover of G.

b) Give a proof for the following fact. Let T be a tree with r vertices that are not a leaf and ℓ leaves. Then there is a matching in T with at least $r/2$ edges. (Hint: use induction.)

c) Show that G has a matching of size at least $|C|/2$.

d) Prove that the algorithm outputs a vertex cover whose size is at most two times the size of a minimum vertex cover.

e) The algorithm above was stated for connected graphs. What would you do for graphs that are not connected?

2. Asymmetric TSP (2.5 points)

Suppose we have an instance of TSP that is not symmetric but still fulfils the triangle inequality. For example, consider a TSP for a postman on a bike in a mountainous area. Also suppose that for every two cities i and j, we have that $d(i,j) \leq 2 * d(j,i)$, and, of course, $d(j,i) \leq 2 * d(i,j)$.

Give an approximation algorithm for this problem that runs in polynomial time and has a performance ratio bounded by a constant.

3. Modulator to single edges (3.5 points)

Consider the following parameterized problem:

> Modulator to single edges - parameterized version
> **Given**: Graph $G = (V, E)$, integer $k \geq 0$
> **Question**: Is there a set of vertices $W \subseteq V$ such that the graph obtained by removing all vertices in W consists of only vertices of degree 0 and 1 and $|W| \leq k$?
> **Parameter:** k

I.e., when W is removed, the remainder of the graph are single vertices and edges. Call such a set W a *modulator to single edges*.

And consider its optimization version:

> Modulator to single edges - optimization version
> **Given**: Graph $G = (V, E)$
> **Question**: Give a minimum size set of vertices $W \subseteq V$ such that the graph obtained by removing all vertices in W consists of only vertices of degree 0 and 1?
> **Parameter:** k

(i) Let v be a vertex of degree at least two. Let w and x be two of the neighbors of v. Let S be a modulator to single edges. Argue that $\{v, w, x\} \cap S \neq \emptyset$.

(ii) Let v be a vertex of degree at least $k + 2$. Let S be a modulator to single edges with $|S| \leq k$. Argue that $v \in S$.

(ii) Show that Modulator to single edges - parameterized version can be solved in $O(3^k p(n))$ time for some polynomial p, with help of branching.

(iv) Consider the following algorithm.

$S = \emptyset$;
Let $H = G$; *i.e., we copy G to the graph H*;
while H contains at least one vertex of degree at least two **do**
    Take a vertex v of degree at least two in H.
    Take two neighbors w and x of v in H;

Add v, w, and x to S;
Remove v, w and x from H, together with all edges with v, w or x as endpoint;
enddo;
Output S.

Argue that this algorithm is a 3-approximation for the MODULATOR TO SINGLE EDGES - OPTIMIZATION VERSION problem. I.e., the set S that is given as output is at most three times as large as an optimal solution to the problem.

(v) Show that MODULATOR TO SINGLE EDGES - PARAMETERIZED VERSION has a kernel with $O(k^2)$ vertices. *(Hint: modify a method for a similarly looking problem, and show that this works correctly.)*

4. Large cluster subgraph (1.5 point)

Consider the following problem:

LARGE CLUSTER SUBGRAPH
Given: Undirected graph $G = (V, E)$, integer K.
Question: Is there a set $W \subseteq V$ of at most K vertices (i.e., $|W| \leq K$), such that G is a cluster graph when we delete all vertices in W and their edges from G, i.e., $G[V \setminus W]$ is a cluster graph?

Prove that the LARGE CLUSTER SUBGRAPH problem is fixed parameter tractable.

Bonus Question: No Guarantees for Greedy Vertex Cover (1 bonus point)

Someone proposes the following algorithm to approximate the VERTEX COVER problem.

Start with an empty vertex set $C = \emptyset$.
While there edges that are not covered by C:
Add the vertex in G incident to the most uncovered edges to C.

Show that this algorithm is not a c-approximation for any fixed value of c.